DECOMPOSITIONS THAT DESTROY SIMPLE CONNECTIVITY

BY

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We shall be concerned with a monotone decomposition of $R^3$ with only one nondegenerate decomposition element $X$. We use $g$ to denote the decomposition map and $g(R^3)$ the decomposition space. Also, $D$ denotes a disk. To determine if $g(R^3)$ is simply connected we shall be concerned with whether maps of $\text{Bd } D$ into $g(R^3)$ can be extended to $D$.

At the Summer Institute on Set Theoretic Topology at Wisconsin in 1955 I gave a talk entitled “What topology is here to stay” in which I envisioned decompositions of $R^3$ as a very viable area for research. I mentioned R.L. Moore’s monotone decomposition theorem [3] for $S^2$ which states that if $G$ is a nondegenerate upper semicontinuous decomposition of $S^2$ each of whose elements is a continuum that does not separate $S^2$, then the decomposition space is $S^2$. I pointed out that the theorem was false if one replaced $S^2$ by $S^3$ and gave as an example the decomposition whose only nondegenerate element is a circle. The earlier version of the Summary of Lectures and Seminars [1] reported on page 26 that the reason I gave that the decomposition space differed from $S^3$ was that it is not simply connected. The second printing of [1] made the correction by replacing the is not simply connected part of the statement by does not remain simply connected on the removal of some point. It was also claimed there and in [2] that the decomposition space of $S^3$ (or $R^3$) whose only nondegenerate element is a solenoid is not simply connected. When I was assembling copies of my publications it was called to my attention that a proof of this claim had not been published. It is the purpose of this paper to fill that gap. Other claims were made in [2] about the simple connectivity of other monotone decompositions (perhaps with many nondegenerate elements) of $R^3$, but we shall not treat them in this paper.

Richard Skora read an early draft of this paper and made valuable suggestions for improving some proofs.

1. $X$ is a standard solenoid

In this case $X$ is the intersection of smooth unknotted tori $T_1, T_2, \ldots$ where $T_{i+1}$ winds around $T_i$ smoothly more than once, the meridional cross sections

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of \( T_{i+1} \) are round planar disks that lie in those of \( T_i \), and the diameters of the meridional cross sections of the \( T_i \)'s converge to 0 as \( i \) increases without limit. Sometimes the restriction of "more than once" is omitted and a circle is permitted to be a solenoid—but we shall not do that in this paper.

**Theorem 1.** If \( X \) is a standardly embedded solenoid, \( g(R^3) \) is neither simply connected nor locally simply connected.

**Proof.** If we seek a map \( f \) of \( Bd D \) into \( g(R^3) \) that cannot be extended to \( D \), we should seek one such that \( g(X) \in f(Bd D) \) because if \( g(X) \notin f(Bd D) \), there is an extension of \( g^{-1}f \) on \( Bd D \) to take \( D \) into \( R^3 \). This extension followed by \( g \) would extend \( f \) to map \( D \) into \( g(R^3) \).

Let \( pq \) be an arc in a meridional cross section of \( T_1 \) that intersects \( X \) only in its end points where these end points belong to different arc components of \( X \). We show that \( g(R^3) \) is not simply connected by showing that a homeomorphism \( f \) of \( Bd D \) onto \( g(pq) \) cannot be extended to map \( D \) into \( g(R^3) \). Since for each open subset \( U \) of \( g(R^3) \) containing \( g(X) \), there is a \( pq \) with \( g(pq) \) in \( U \), this will also show that \( g(R^3) \) is not locally simply connected. See Figure 1.

Assume \( f \) is a homeomorphism of \( Bd D \) onto \( g(pq) \) and that \( f \) can be extended to a map \( F \) of \( D \) into \( g(R^3) \). It would be nice if \( F^{-1}(g(X)) \) were 0-dimensional, so we adjust \( F \) to get a new map \( F_2 \) where \( F_2^{-1}(g(X)) \) is 0-dimensional. First, let \( F_1 \) be a map of \( D \) into \( g(R^3) \) such that \( F_1 = F \) on the component of \( D - F^{-1}(g(X)) \) intersecting \( Bd D \) and \( F_1 \) takes the rest of \( D \) to \( g(X) \). Next we let \( k \) be a map of \( D \) onto itself that is the identity on \( Bd D \) and whose point inverses are the components of \( F_1^{-1}(g(X)) \) and points of \( D - F_1^{-1}(g(X)) \). Moore's decomposition theorem [3] mentioned earlier is used to get \( k \). Then \( F_2 = F_k^{-1} \). For simplicity we suppose \( F = F_2 \).

Let \( a_0b_0 \) be a spanning arc of \( D \) such that \( F(a_0b_0) \) misses \( g(X) \) and the subdisk \( D_0 \) of \( D \) bounded by the union of \( a_0b_0 \) and the subarc of \( Bd D \) from \( a_0 \) to \( b_0 \) through \( f^{-1}(g(X)) \) lies in \( F^{-1}(g(\text{Int} T_1)) \). Since \( g^{-1}(F(a_0)) \) and \( g^{-1}(F(b_0)) \) lie in the same meridional cross section of \( T_1 \), we can speak of the number of times that \( g^{-1}F(a_0b_0) \) winds around \( T_1 \).

For some large \( r \) let \( a_rb_r \) be a spanning subarc of \( D_0 \) such that \( a_r \) lies on \( Bd D \) between \( a_0 \) and \( f^{-1}g(X) \), \( b_r \) lies on \( Bd D \) between \( b_0 \) and \( f^{-1}g(X) \), \( F(a_rb_r) \) misses \( g(X) \), and \( F(a_rb_r) \) lies in \( g(T_r) \). Let \( D_r \) be the subdisk of \( D_0 \) bounded by union of \( a_0b_0 \), \( a_rb_r \), and two subarcs of \( Bd D \). See Figure 1.

Since \( p \) and \( q \) belong to different arc components of \( X \), for large \( r \), \( g^{-1}F(a_rb_r) \) winds around \( T_1 \) many times—even more than \( g^{-1}F(a_0b_0) \) does. Let \( y(r) \) be the number of times that \( g^{-1}(F(Bd D_r)) \) winds around \( T_1 \).

Suppose \( r \) is so large that \( y(r) > 0 \).

Let \( z(s) \) be the number of times that \( T_s \) winds around \( T_1 \). We suppose \( s \) is so large that \( z(s) > y(r) \) and \( g^{-1}F(Bd D_r) \) misses \( Bd T_r \). We suppose that on \( D_r \) near \( F^{-1}(g(Bd T_s)) \), \( F \) has enough general position so that \( D \cap F^{-1}g(Bd T_s) \) is the union of a finite number of mutually disjoint simply closed
curves $J_1, J_2, \ldots, J_n$. Since each $g^{-1}F(J_i)$ lies on $\text{Bd } T_i$, it winds around $T_1$ some integral multiple of $z(s)$.

Let $E_r$ be the finitely holed $D_r$ obtained by deleting from $D_r$ the interiors of the subdisks of $D_r$ bounded by the $J_i$'s. We now come to the contradiction caused by the assumption that $f$ on $\text{Bd } D$ could be extended to $F$ on $D$. The boundary of each of the holes of $E_r$ winds around $T_1$ some integral multiple of $z(s)$, but $y(r)$ is not an integral multiple of $z(s)$.

2. X is an embedded solenoid

The complement of an embedded solenoid may be quite different from the complement of a standardly embedded solenoid. We no longer can speak of tori about the embedded solenoid. However, we still find that the decomposition space is not simply connected.

**Theorem 2.** If $X$ is an embedded solenoid, $g(R^3)$ is neither simply connected nor locally simply connected.
Proof. We use $X'$ to denote the standardly embedded solenoid of Theorem 1, $p'q'$ the arc called $pq$ there, and $g'$ the decomposition map called $g$ there. Let $\beta$ be a homeomorphism of $X$ onto $X'$ and $pq$ an arc from $\beta^{-1}(p')$ to $\beta^{-1}(q')$ in $\mathbb{R}^3$ that intersects $X$ only at $p$ and $q$. This is possible since the dimension of $X$ is 1. Extend $\beta$ to a map of $\mathbb{R}^3$ onto itself that takes $pq$ homeomorphically to $p'q'$. For convenience, call the extension $\beta$. It is a map rather than a homeomorphism. We finish the proof of Theorem 2 by showing that $g(pq)$ cannot be shrunk to a point in $g(\mathbb{R}^3)$.

Let $f$ be a homeomorphism of $\text{Bd} \ D$ onto $g(pq)$. Assume that $f$ can be extended to map $F$ taking $D$ into $g(\mathbb{R}^3)$. This leads to the contradiction that a homeomorphism of $\text{Bd} \ D$ onto $g'(p'q')$ can be extended to a map $g\beta g^{-1}F$ of $D$ into $g'(\mathbb{R}^3)$. See Figure 2. The proof of Theorem 1 showed that no homeomorphism of $\text{Bd} \ D$ onto $g'(p'q')$ could be extended to map $D$ into $g'(\mathbb{R}^3)$.

![Figure 2](image-url)
Since pq could be picked close to X, this also shows that \( g(X) \) is not locally simply connected.

3. \( X \) is unlike-a-solenoid

We say that a disk \( D \) can be converted to a disk with finitely many holes \( E \) if there is a finite collection of mutually disjoint disks in \( \text{Int} \ D \) and \( E \) is obtained from \( D \) by removing the interiors of these subdisks. These interiors are called holes in \( D \) and \( E \) is called a finitely holed \( D \). We call \( D \) a finitely holed \( D \) even if there are no holes and \( D = E \).

Let \( N_j \) be the \( 1/i \)-neighborhood of \( X \) in \( R^3 \)—that is, the set of points of \( R^3 \) whose distance from \( X \) is less than \( 1/i \).

If \( f(\text{Bd} \ D) \subset N_j - X \), we say that \( f \) can be pulled in \( N_j - X \) arbitrarily close to \( X \) if for arbitrary large \( s \), \( f \) can be extended to take a finitely holed \( D \) into \( N_j - X \) so that the boundary of each hole is sent into \( N_j \). Note that \( s \) is picked before the extension. If a different \( s \) had been chosen, we might have needed a different extension. (If \( f \) can be extended to map \( D \) into \( N_j \), we could have picked an extension independent of \( s \). If \( f \) can be extended to map \( D \) into \( N_j - X \), then technically the definition says that \( f \) can be pulled arbitrarily close to \( X \) even though \( f(D) \) misses \( X \). This is because we call \( D \) a finitely holed \( D \).) We say that \( X \) is unlike-a-solenoid if for each \( N_j \), there is an \( N_j(i) \) such that each map \( f \) of \( \text{Bd} \ D \) into \( N_j(i) - X \) can be pulled in \( N_j(i) - X \) arbitrarily close to \( X \).

**Theorem 3.** If \( X \) is unlike-a-solenoid, then \( g(R^3) \) is simply connected and locally simply connected.

**Proof.** We first show that \( g(R^3) \) is locally simply connected at \( g(X) \). We show that if \( U \) is a neighborhood in \( g(R^3) \) of \( g(X) \), there is a neighborhood \( V \) of \( g(X) \) such that each map \( f \) of \( \text{Bd} \ D \) into \( V \) can be extended to take \( D \) into \( U \). We use \( f \) as a map of \( \text{Bd} \ D \) into \( V \) and \( g^{-1}f \) to send \( \text{Bd} \ D \) into \( R^3 \).

To show that \( g(R^3) \) is locally simply connected, without loss of generality we pick \( U \) to be \( g(N_j) \) and \( V \) to be \( g(N_{r(j)}) \) where \( r(j) \) is an integer such that any map of \( \text{Bd} \ D \) into \( N_{r(j)} - X \) can be pulled in \( N_j - X \) arbitrarily close to \( X \) on a finitely holed \( D \).

We consider the sequence \( n_1, n_2, \ldots \) where \( n_1 = j \), and \( n_{i+1} = r(n_i) \). Although the \( r \)'s are defined for maps of \( \text{Bd} \ D \) into \( R^3 - X \), we realize that in the case of the \( f \) of \( \text{Bd} \ D \) into \( V = g(N_{r(j)}) \) we wish to extend it to a map from \( D \) into \( U = g(N_j) \) and this \( f(\text{Bd} \ D) \) may contain \( g(X) \).

It may be that \( f^{-1}(g(X)) \) has several components. We wish to avoid this. With that purpose in mind we suppose \( D \) is a round planar disk and let \( C \) be the convex hull of \( f^{-1}(g(X)) \) and partially extend \( f \) to send \( C \) to \( g(X) \). For convenience we call the extension \( \tilde{f} \). Now \( f \) is defined except on a collection
(possibly infinite) of open disks. We call these disks $D_i$'s. For each of these disks, $f$ sends the straight subarc of its boundary to $g(X)$ and the open curved part into $g(R^3 - X)$. We have simplified the situation so that $f^{-1}g(X)$ intersects each $\text{Bd } D_i$ in a connected set. See Figure 3.

Let $D_i$ be one of the subdisks on whose interior $f$ has not been defined. Each $\text{Bd } D_i \subset g(N_{n_2})$ and for all but a finite number of these $D_i$'s $f(\text{Bd } D_i) \subset g(N_{n_3})$. For this finite number where $f(\text{Bd } D_i) \not\subset g(N_{n_3})$, we extend $f$ to a part of their interiors.

Let $ab$ be a spanning arc of $D_i$ that cuts $D_i$ into two subdisks $D'_i$, $D''_i$ where $f(\text{Bd } D'_i - ab)$ misses $g(X)$ and the closure of $f(\text{Bd } D''_i - ab)$ lies in $g(N_3)$. Extend $f$ to take $ab$ into $g(N_{n_3} - X)$. Extend $f$ further on a finitely holed $D'_i$ to take the finitely holed $D'_i$ into $g(N_{n_1} - X)$ where the boundary of the holes go into $g(N_{n_3})$. To get the extension we use the hypothesis that $X$ is unlike-a-solenoid.
We now find that \( f \) is defined except on open disks whose boundaries are sent by \( f \) into \( g(N_{n_1}) \). In fact on all but a finite number of these disks, \( f \) sends their boundaries into \( g(N_{n_2}) \). We extend \( f \) into a part of the interiors of this finite collection so that now \( f \) is defined except on open disks whose boundaries are sent into \( g(N_{n_2}) \). Using the hypothesis that \( X \) is unlike-a-solenoid we pick the extension on the new part to take this new part into \( g(N_{n_2}) \).

The extension is extended a countable number of times and finally we define the extension to take the remaining part of \( D \) to \( g(X) \). We have now shown that \( g(R^3) \) is locally simply connected. That it is simply connected follows from the following theorem.

**Theorem 4.** If \( g(R^3) \) is locally simply connected, it is simply connected.

*Proof.* Let \( f \) be a map of \( Bd \ D \) into \( g(R^3) \). We show that \( f \) is simply connected by showing that \( f \) can be extended to map \( D \) into \( g(R^3) \).

It follows from the local simple connectivity of \( g(R^3) \) that there is a neighborhood \( U \) of \( g(X) \) such that each map of \( Bd \ D \) into \( U \) can be extended to map \( D \) into \( g(R^3) \). Suppose \( D \) is a round disk and \( A_1, A_2, \ldots, A_n \) are the components of \( Bd \ D - f^{-1}g(X) \) that are not sent into \( U \) by \( f \). Let \( B_i \) be an open arc on \( A_i \) such that each \( f(A_i - B_i) \subset U \). Let \( C_i \) be a straight arc in \( D \) joining the two components of \( A_i - B_i \).

Extend \( f \) to \( C_i \) so that the extension takes \( C_i \) into \( U - g(X) \). Call the extension \( f \). On the subdisk \( D_i \) of \( D \) bounded by \( C_i \) and a part of \( A_i \), extend \( g^{-1}f \) to take each \( D_i \) into \( R^3 \) and follow this extension by \( g \) to extend \( f \) to take \( D_i \) into \( g(R^3) \). The boundary of the remaining part of \( D \) is sent by \( f \) into \( U \) so the local connectivity of \( g(R^3) \) shows that \( f \) can be extended to the rest of \( D \). This shows that \( g(R^3) \) is simply connected.

4. \( X \) is solenoid-like

We say that \( X \) is solenoid-like if there is a neighborhood \( N \) of \( X \) in \( R^3 \) such that for any neighborhood \( N' \) of \( X \) there is a map \( f \) of \( Bd \ D \) into \( N' - X \) which cannot be pulled in \( N - X \) on a finitely holed \( D \) arbitrarily close to \( X \). One might note if \( X \) is solenoid-like, then it is untrue that \( X \) is unlike-a-solenoid.

**Theorem 5.** If \( X \) is solenoid-like, \( g(R^3) \) is not locally simply connected.

*Proof.* We show that for some neighborhood \( U = g(N) \) of \( g(X) \), and each smaller neighborhood \( V = g(N') \) of \( g(X) \) there is a map of \( Bd \ D \) into \( V \) that cannot be extended to map \( D \) into \( U \). Here we use \( N, N', f \) as in definition of solenoid-like and use \( gf \) for the map of \( Bd \ D \) into \( V - g(X) \) that cannot be
extended to map $D$ into $U$. If $gf$ could be extended by $F$ to send $D$ into $U$, $g^{-1}F$ would show that $f$ can be pulled in $N - X$ on a finitely holed $D$ arbitrarily close to $X$.

**Question** Recall that $X$ is a continuum in $R^3$ and $g(R^3)$ is the decomposition space whose only nondegenerate point inverse is $X$. Is $g(R^3)$ locally simply connected if it is simply connected? If $g(R^3)$ is not locally simply connected, could it be simply connected?

### 5. Necessary and sufficient conditions

Theorems 3 and 5 provide a necessary and sufficient condition that $g(R^3)$ is not locally simply connected. However the condition is dependent on the embedding of $X$ and does not say that if $X'$ is homeomorphic to $X$ then $g(R^3) = R^3/X$ is locally connected if and only if $R^3/X'$ is.

**Theorem 6.** A necessary and sufficient condition that $g(R^3)$ not be locally simply connected is that $X$ be solenoid-like.

**Proof.** The sufficiency is provided by Theorem 5 and the necessity by Theorem 3.

**Theorem 7.** If $X$ and $X'$ are homeomorphic continua in $R^3$ and dim $X = 1$, then $g(R^3) = R^3/X$ fails to be locally simply connected if and only if $g'(R^3) = R^3/X'$ does.

**Proof.** The proof of Theorem 7 is modelled after that of Theorem 2.

Suppose $g'(R^3)$ is not locally simply connected. Then it is solenoid-like and there are a neighborhood $N'$ of $X'$ and a sequence of mutually disjoint simple closed curves $J'_1, J'_2, \ldots$ in $N' - X'$ such that $J'_i$ lies in the $1/i$ neighborhood of $X'$ and a map $f'_i$ of $Bd D$ onto $J'_i$ that cannot be pulled in $N' - X'$ arbitrarily close to $X'$ on a finitely-holed $D$.

Let $\beta$ be a homeomorphism of $X$ onto $X'$. We now pick a sequence of simple closed curves $J_1, J_2, \ldots$ in $R^3 - X$ so that $\beta$ can be extended to a homeomorphism taking $X \cup J_1 \cup J_2 \cup \ldots$ onto $X' \cup J'_1 \cup J'_2 \cup \ldots$. We assume $J_1, J_2, \ldots, J_{i-1}$ have been found with $\beta$ extended to them and describe $J_i$ and $\beta$ on it.

Express $J'_i$ as the union of arcs $a'_1a'_2, a'_2a'_3, \ldots, a'_{n}a'_{n+1}$ each of diameter less than $1/i$. Let $b'_i$ be a point of $X'$ in the $1/i$ neighborhood of $a'_i$. Note that the distance between two adjacent $b'_j$’s is less than $3/i$. Let $a_j$ be a point of

$$R^3 - (X \cup J_1 \cup \cdots \cup J_{i-1})$$
in the $1/i$ neighborhood of $\beta^{-1}(a_j)$. Note that distance between two adjacent $a_j$'s is less than $\varepsilon + 2/i$ where $\varepsilon$ is a positive number such that the image under $\beta^{-1}$ of any $3/i$-subset of $X'$ has diameter less than $\varepsilon$. Hence adjacent $a_j$'s are close if $i$ is large. We suppose the $a_j$'s are distinct and let $J_i$ be a simple closed curve in

$$R^3 - (X \cup J_1 \cup J_2 \cup \cdots \cup J_{i-1})$$

which is the union of arcs $a_1a_2, a_2a_3, \ldots, a_na_{n+1}$ where $a_ia_{i+1}$ lies in the $1/i$ neighborhood of the straight line interval from $a_i$ to $a_{i+1}$. It is in showing that there are such $a_ia_{i+1}$'s in $R^3 - (X \cup J_1 \cup J_2 \cup \cdots \cup J_{i-1})$ that we use the fact that $\dim X = 1$. The homeomorphism $\beta$ is extended to $J_i$ so that $\beta(a_j) = a'_j$.

Suppose all $J_i$'s are defined. Call the extended homeomorphism $\beta$. Now extend $\beta$ to a map taking $R^3$ onto $R^3$ and call this extension $\beta$ also. This final extension need not be a homeomorphism. Let $f_i$ be a homeomorphism of $\text{Bd} D$ onto $J_i$ such that $\beta f_i = f'_i$.

Assume that $g(R^3) = R^3/X$ is locally simply connected. We prove the theorem by showing that this assumption is false. Local simple connectivity of $g(R^3)$ implies that for a large $r$, $gF_r$ on $\text{Bd} D$ can be extended to a map $F_r$ taking $D$ into $g\beta^{-1}(N')$. Let $C$ be the set of points of $D$ that are carried by $\beta g^{-1}F_r$ onto $X'$ or points of $X'$. Change $D$ to $E$, a finitely holed $D$ in $D - C$, so that the boundaries of the holes of $E$ are very close to $C$. Although $\beta g^{-1}F_r$ need not be a map on $D$, it is one on $E$. Also $\beta g^{-1}F_r = f'_r$ on $\text{Bd} D$. Hence $\beta g^{-1}F_r$ on $E$ shows that $f'_r$ on $\text{Bd} D$ can be pulled in $N' - X'$ arbitrarily close to $X$ on a finitely holed $D$. This contradicts the selection of $J'_i$ and $f'_i$.

REFERENCES

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