VIETORIS-BEGLE THEOREM AND SPECTRA

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Abstract. The following generalization of the Vietoris-Begle Theorem is proved: Suppose \( \{E_k\}_{k \geq 1} \) is a CW spectrum and \( f: X' \to X \) is a closed surjective map of paracompact Hausdorff spaces such that \( \text{Ind } X = m < \infty \).

If \( f^*: E^k(\mathcal{U}_k) \to E^k(f^{-1}(\mathcal{U}_k)) \) is an isomorphism for all \( x \in X \) and \( k = m_0, m_0 + 1, \ldots, m_0 + m \), then \( f^*: E^n(X) \to E^n(X') \) is an isomorphism and \( f^*: E^{n+1}(X) \to E^{n+1}(X') \) is a monomorphism for \( n = m_0 + m \).

Given a CW spectrum \( E = \{E_k\}_{k \geq 1} \) and a pointed CW complex \( K \), one has cohomology groups \( E^n(K) \) for each integer \( n \) (see [Sw, Chapter 8]). They are defined as homotopy classes from the suspension spectrum of \( K \) to \( \Sigma^n E \), where \( \Sigma^n E \) is defined by \( \Sigma^n E_k = E_{k+n} \). In the case of an \( \Omega \)-spectrum (i.e., where the natural map \( E_k \to \Omega E_{k+1} \) is a homotopy equivalence for all \( k \)), \( E^n(K) \) is isomorphic to \( [K, E_n] \) (see [Sw, Theorem 8.42]). The groups \( E^n(X), X \) being any pointed topological space, are defined as \( \text{dirlim}\{E^n(X_\alpha), p_{\alpha\beta}^*, \Lambda\} \), where \( \{X_\alpha, p_{\alpha\beta}, \Lambda\} \) is the Čech system of \( X \) (see [D-S, p. 21] for the definition of the Čech system of \( X \)). In this way one gets the Čech extension of the functor \( E^n \) from pointed CW complexes to pointed spaces (see [D] for a general discussion of Čech extensions of functors). Again, if \( \{E_k\}_{k \geq 1} \) is an \( \Omega \)-spectrum, then \( E^n(X) \) is isomorphic to \( [X, E_n] \). A basic result is that every spectrum \( \{E_k\}_{k \geq 1} \) is isomorphic to an \( \Omega \)-spectrum (see [B, part 10 of Chapter II]). Essentially, the \( n \)th term of that spectrum is the telescope of \( E_n \to \Omega E_{n+1} \to \Omega^2 E_{n+2} \to \cdots \).

In the case of an unpointed topological space \( X \), we define the unreduced cohomology \( E^n(X) \) as \( E^n(X^+) \), where \( X^+ \) is \( X \) with a discrete base point added.

Recall the classical Vietoris-Begle Theorem (see [S, p. 344]):

**Vietoris-Begle Theorem.** Let \( f: X' \to X \) be a closed surjective map of paracompact Hausdorff spaces. Assume that there is an \( n \geq 0 \) such that \( \bar{H}^k(f^{-1}(x)) = 0 \) (reduced Čech cohomology) for all \( x \in X \) and for \( k < n \). Then \( f^*: H^k(X) \to H^k(X') \) is an isomorphism for \( k < n \) and a monomorphism for \( k = n \).

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A natural question arises: Since Čech cohomology corresponds to the Eilenberg-MacLane spectrum \( \{K(G, k)\}_{k \geq 1} \), is there a generalization of the above theorem to arbitrary spectra?

The naive approach of replacing \( H^k(f^{-1}(x)) = 0 \) by \( E^k(f^{-1}(x)) \approx E^k(\{x\}) \) does not work. An example of that is Taylor's cell-like map (see [T]) \( f: X \to Q \) onto the Hilbert cube \( Q \) such that \( \tilde{K}(X) \) is nonzero (\( \tilde{K} \) is the reduced complex \( K \) theory which is the cohomology theory of the spectrum \( BU \), the periodic spectrum \( U, BU, \ldots \)). This example was later modified by J. Keesling [K] who produced a cell-like map \( g: Q \to Y \) with \( \tilde{K}(Y) \neq 0 \).

The aim of this paper is to prove suitable generalizations of the Vietoris-Begle Theorem involving arbitrary unreduced spectral cohomology. In this task we were guided by Kozlowski's result (see [Ko] or [D-S]) proved in 1975.

**Theorem (G. Kozlowski).** For closed maps \( f: X \to Y \) of metrizable spaces such that \( [f^{-1}(A), K] = [A, K] \) for any CW complex \( K \) and any closed subset \( A \) of \( Y \), the image \( Y \) is an ANR provided \( X \) as an ANR.

Kozlowski's proof involved a trick: he showed that certain maps from \( X \) to \( K \) can be extended over the mapping cylinder \( M(f) \) of \( f \). Then he replaced \( X \) by the double mapping cylinder \( DM(f) \) of \( f \) (the union of two copies of \( M(f) \) sewn along \( X \)) and used the previous construction to relate any two different extensions. The meaning of this trick is that it echoes the Mayer-Vietoris exact sequence (once you prove that a certain homomorphism is onto you get that the next one is trivial, for free). In this paper we formalize this observation: the trick becomes Lemma 3 (the mapping cone \( C_p \) of \( p: DM(f) \to Y \) is homeomorphic to the reduced suspension \( S(C_f) \) of the mapping cone of \( f \)) and the whole approach resembles a Puppe exact sequence. Notice that Theorem B implies the results of [D-K] (the goal of that paper was to unify Vietoris-Begle Theorem and cell-like maps on spaces of finite deformation dimension).

**Theorem A.** Suppose \( \{E_k\}_{k \geq 1} \) is a CW spectrum and \( f: X' \to X \) is a closed surjective map of paracompact Hausdorff spaces such that \( \text{Ind} X = m < \infty \). If \( f^*: E^n(x) \to E^n(f^{-1}(x)) \) is an isomorphism for all \( x \in X \) and \( n = m_0, m_0 + 1, \ldots, m_0 + m \), then \( f^*: E^n(X) \to E^n(X') \) is an isomorphism and \( f^*: E^{n+1}(X) \to E^{n+1}(X') \) is a monomorphism for \( n = m_0 + m \).

**Remark.** \( \text{Ind} X \) is the large inductive dimension of \( X \): \( \text{Ind} \emptyset = -1 \) and \( \text{Ind} X \leq m \) means that for any neighborhood \( U \) of a closed subset \( A \) of \( X \), there is a neighborhood \( V \) of \( A \) in \( U \) with \( \text{Ind}(\text{cl}(V) - V) \leq m - 1 \).

**Theorem B.** Suppose \( \{E_k\}_{k \geq 1} \) is a CW spectrum and \( f: X' \to X \) is a closed surjective map of paracompact Hausdorff spaces such that the following conditions are satisfied:

(a) \( f^*: E^{n-1}(A) \to E^{n-1}(f^{-1}(A)) \) is an epimorphism for all closed subsets \( A \) of \( X \), and
(b) \( f^* : E^n(A) \to E^n(f^{-1}(A)) \) is a monomorphism for all closed subsets \( A \) of \( X \).

If \( f^* : E^n(x) \to E^n(f^{-1}(x)) \) is an isomorphism for all \( x \in X \), then \( f^* : E^n(X) \to E^n(X') \) is an isomorphism and \( f^* : E^{n+1}(X) \to E^{n+1}(X') \) is a monomorphism.

**Remark.** Obviously, conditions (a) and (b) are derived from Kozlowski’s Theorem.

The proofs of Theorems A and B will depend on Lemmas 1–4 below.

Given a map \( f : X' \to X \) and a subset \( A \) of \( X \), \( f^{-1}(A) \) is denoted by \( A' \) and the map \( A' \to A \) defined by \( f \) is denoted by \( f_A \).

Let \( C_f = M(f)/X' \) be the mapping cone of a map \( f : X' \to X \). \( q_f : M(f) \to C_f \) denotes the quotient map from the mapping cylinder \( M(f) \) of \( f \) to \( C_f \).

Given a map \( f : X' \to X \) and a space \( E \), \( f^* : [X, E] \to [X', E] \) is called monic provided that for any map \( g : X \to E \), \( gf \equiv \text{const} \) implies \( g \equiv \text{const} \).

**Lemma 1.** Suppose \( f : X' \to X \) is a map such that \( f^* : [X, \Omega E] \to [X', \Omega E] \) is onto. Then \( q_f^* : [C_f, E] \to [M(f), E] \) is monic.

**Proof.** Suppose \( g : C_f \to E \) is a map such that \( g|X \approx \text{const} \). We may assume \( g|X = \text{const} \) (by homotoping \( g \)). Then \( g \) factors as \( C_f \to \Sigma X' \to E \), which in turn factors (up to homotopy) as \( C_f \to \Sigma X' \to \Sigma X \to E \). Notice that \( C_f \to \Sigma X \) is null-homotopic, as it factors as \( C_f \to C(X) \to \Sigma X \), \( C(X) \) being the cone over \( X \). Thus \( g \approx \text{const} \).

**Lemma 2.** Suppose \( f : X' \to X \) is a map such that \( q_f^* : [C_f, E] \to [M(f), E] \) is monic. If \( g, h : M(f) \to E \) are two null-homotopic maps such that \( g|X' = h|X' \), then \( g \approx h \) rel. \( X' \).

**Proof.** We need to extend the map \( G : M(f) \times \{0, 1\} \cup X' \times I \to E \), where \( G|M(f) \times \{0\} = g \), \( G|M(f) \times \{1\} = h \), and \( G(x, t) = g(x) \) for \( (x, t) \in X' \times I \) over \( M(f) \times I \). Since \( G|X \times \{0\} \approx \text{const} \), \( G \) extends to \( G' : (M(f) \times \{0, 1\} \cup X' \times I) \cup C(X \times \{0\}) \to E \), where \( C(X \times \{0\}) \) is the cone over \( X \times \{0\} \). Notice that \( M(f) \times \{0, 1\} \cup X' \times I \) \( \cup C(X \times \{0\}) \) is homotopy equivalent to \( C_f \) and \( G'|X \times \{1\} \) is null-homotopic. By the hypotheses, \( G' \approx \text{const} \), which implies \( G \approx \text{const} \). Since the pair \( (M(f) \times I, M(f) \times \{0, 1\} \cup X' \times I) \) has the homotopy extension property with respect to any space, we obtain an extension of \( G \) over \( M(f) \times I \).

Recall that the double mapping cylinder \( DM(f) \) of a map \( f : X' \to X \) is the union of two copies of \( M(f) \) with two copies of \( X' \) identified. The natural projection \( DM(f) \to X \) is denoted by \( p \).

**Lemma 3.** For any map \( f : X' \to X \), the mapping cone \( C_p \) of the natural projection \( p : DM(f) \to X \) is homeomorphic to the reduced suspension \( S(C_f) \) of the mapping cone of \( f \).
Proof. Notice that $\text{DM}(f)$ is homeomorphic to $X' \times I \cup M(f) \times \{0, 1\} \subset M(f) \times I$, and $M(p)$ is homeomorphic to $M(f) \times I$. Also $C_p$ is homeomorphic to $M(f) \times I / (X' \times I \cup M(f) \times \{0, 1\})$. Since $\Sigma(C_f) = (M(f)/X') \times I / (M(f)/X') \times \{0, 1\}$, Lemma 3 follows.

Lemma 4. Suppose $E$ is a CW complex and $f: X' \to X$ is a closed surjective map of paracompact Hausdorff spaces. Denote by $\mathcal{S}$ the family of all closed subsets $B$ of $X$ such that $q_f^*: [C_f, E] \to [M(f), E]$ and $f_A^*: [A, E] \to [f^{-1}(A), E]$ are monic for any closed subset $A$ of $B$. If for any closed subset $B$ of $X$ and for any neighborhood $U$ of $B$ there is an open neighborhood $V$ of $B$ in $U$ such that $\text{cl}(V) - V \in \mathcal{S}$, then the image of $f^*: [X, E] \to [X', E]$ is precisely the set of all homotopy classes $[g]$ such that $g|f^{-1}(x) \approx \text{const}$ for all $x \in X$.

Proof. It suffices to show that any map $g: X' \to E$ such that $g|f^{-1}(x) \approx \text{const}$ for all $x \in X$ extends over $M(f)$. Without loss of generality we may assume that $E$ is an ANE for paracompact spaces (see [D-K]). Let $\pi: M(f) \to X$ be the projection. Fix $x \in X$. Since $g|f^{-1}(x) \approx \text{const}$, there exists an extension $g': X' \cup \pi^{-1}(x) \to E$ of $g$. Define $g'': X' \cup \pi^{-1}(x) \cup X \to E$ by $g''|X' \cup \pi^{-1}(x) = g'$ and $g''(X) = g(x)$; $g''$ extends over a neighborhood $U$ of $X' \cup \pi^{-1}(x) \cup X$ in $M(f)$. Choose a neighborhood $V_x$ of $x$ in $X$ such that $\pi^{-1}(V_x) \subset U$. Having done that for all $x$ in $X$, we choose a locally finite cover $\{A_s : s \in S\}$ of $X$ consisting of closed sets, which is a refinement of $\{V_x : x \in X\}$. Then, for each $s \in S$, we choose a map $g_s: X' \cup \pi^{-1}(A_s) \to E$ such that $g_s|X' = g$ and $g_s(A_s)$ is a one-point set.

If $g': X' \cup \pi^{-1}(A) \to E$ is an extension of $g$ ($A$ closed in $X$) and $s \in S$, then there is an extension $g''$ of $g'$ over $X' \cup \pi^{-1}(U)$ for some closed neighborhood $U$ of $A$. Choose an open neighborhood $V$ of $A$ in $U$ such that $\text{cl}(V) - V \in \mathcal{S}$. By Lemma 2,

$$g''|\pi^{-1}((\text{cl}(V) - V) \cap A_s) \approx g_s|\pi^{-1}((\text{cl}(V) - V) \cap A_s)$$

rel. $f^{-1}((\text{cl}(V) - V) \cap A_s)$. Since $g_s|\pi^{-1}((\text{cl}(V) - V) \cap A_s)$ extends over $\pi^{-1}(A_s)$, $g''$ extends over $\pi^{-1}(A_s)$. Thus we have an extension $g'''': X' \cup \pi^{-1}(A \cup A_s) \to E$ of $g$. By well-ordering $S$ and transfinite induction, we can construct an extension $G: M(f) \to E$ of $g$.

Proof of Theorems A and B. We are going to prove the following statement which implies both Theorems A and B:

(*) Suppose $\{E_k\}_{k \geq 1}$ is a CW spectrum and $f: X' \to X$ is a closed surjective map of paracompact Hausdorff spaces such that, for some integer $n$, $f^*: E^n(x) \to E^n(f^{-1}(x))$ is an isomorphism for all $x \in X$. Denote by $\mathcal{S}$ the family of all closed subsets $B$ of $X$ such that $f^*: E^{n-1}(A) \to E^{n-1}(f^{-1}(A))$ is an epimorphism, and $f^*: E^n(A) \to E^n(f^{-1}(A))$ is a monomorphism for all closed subsets $A$ of $B$. If for any closed subset $B$ of $X$ and for any
neighbourhood $U$ of $B$ there is an open neighborhood $V$ of $B$ in $U$ such that $\text{cl}(V) - V \in \mathcal{S}$, then $f^*: E^n(X) \to E^n(X')$ is an isomorphism and $f^*: E^{n+1}(X) \to E^{n+1}(X')$ is a monomorphism.

Without loss of generality, assume $\{E_k\}_{k \geq 1}$ is an $\Omega$-spectrum. Now, $E^n(Z) = [Z, E_n]$ for any space $Z$.

Notice that $f^*: E^n(X) \to E^n(X')$ is an isomorphism by Lemma 4. Indeed, $f_A^*: [A, E_n] \to [f^{-1}(A), E_n]$ is monic for each $A \in \mathcal{S}$, and Lemma 1 implies that $q^*: [C_f, E_n] \to [M(f_A), E_n]$ is monic for each $A \in \mathcal{S}$.

So it remains to show that $f^*: E^{n+1}(X) \to E^{n+1}(X')$ is a monomorphism. Suppose $g, h: X \to E^{n+1}$ are two maps such that $gf \approx hf$. Then there is a map $H: DM(f) \to E^{n+1}$ such that $H$ restricted to one copy of $X$ equals $g$ and $H$ restricted to the other copy of $X$ equals $h$. It suffices to show that $H$ restricts over $M(p)$, where $p: DM(f) \to X$ is the natural projection. This is easily seen if one notices that $(M(p), DM(f))$ is homeomorphic to $(M(f) \times I, X' \times I \cup M(f) \times \{0, 1\})$. Then $H|X \times \{0\} = g, H|X \times \{1\} = h$, and any extension of $H$ over $M(f) \times I$ would produce a homotopy from $g$ to $h$ when restricted to $X \times I$.

Notice that $H|p^{-1}(x)$ is null-homotopic for all $x \in X$. Indeed, $p^{-1}(x)$ is the suspension $\Sigma f^{-1}(x)$ of $f^{-1}(x)$ and $[\Sigma f^{-1}(x), E_{n+1}] = [f^{-1}(x), \Omega E_{n+1}] = \{f^{-1}(x), E_n\} = \{(x), E_n\}$. To be able to apply Lemma 4, we need to check that for each $A \in \mathcal{S}$, both $q^*: [C_f, E_{n+1}] \to [M(p_A), E_{n+1}]$ and $p^*: [A, E_{n+1}] \to [p^{-1}(A), E_{n+1}]$ are monic. The latter is clear, since $p$ is a retraction. By Lemma 3, $[C_f, E_{n+1}] = [S(f_A), E_{n+1}] = [f_A, E_{n+1}] = [C_f, E_{n+1}]$ (here $C_f$ and $C_f$ are considered as pointed spaces with obvious base points). The proof can be completed by showing that any map $u$ from $C_f$ to $E_n$ is null-homotopic (this implies that all pointed maps from $C_f$ to $E_{n+1}$ are null-homotopic, and since all components of $E_{n+1} \cong \Omega E_{n+2}$ are of the same homotopy type, all unpointed maps from $C_f$ to $E_{n+1}$ are null-homotopic). Since $(u|A)f \approx \text{const}$, $u|A \approx \text{const}$ by the first part of the Theorem. Thus $u$ factors (up to homotopy) as $C_f \to C_f/\sim \to E_n$. Since $f^*: E^{n+1}(A) \to E^{n+1}(f^{-1}(A))$ is an epimorphism, the map $\Sigma f^{-1}(A) \to E_n$ factors (up to homotopy) as $\Sigma f^{-1}(A) \to \Sigma(A) \to E_n$. Since $C_f \to \Sigma(A)$ factors through the cone over $A$, $u \approx \text{const}$.

Statement (*) obviously implies Theorem B. Theorem A can be proved by induction on $m = \text{Ind} X$. If $m = 0$, we use Statement (*) with $\mathcal{S}$ being empty and $n = m_0$. If Theorem A holds for $m \leq k$ and $\text{Ind} X = k + 1$, we use Statement (*) with $\mathcal{S} = \{B \in \text{cl}(B) \subset X| \text{Ind} B \leq k\}$ and $n = k + 1 + m_0$.

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References


