A CHARACTERIZATION OF ABSOLUTE NEIGHBORHOOD RETRACTS

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By an absolute neighborhood retract (ANR) I mean a separable metrizable space which is a neighborhood retract of every separable metrizable space which contains it and in which it is closed. This generalization of Borsuk's original definition\(^1\) was given by Kuratowski\(^2\) for the purpose of enlarging the class of absolute neighborhood retracts to include certain spaces which are not compact. The space originally designated by Borsuk as absolute neighborhood retracts (or $K$-sets) will now be referred to as compact absolute neighborhood retracts. Many of the properties of compact ANR-sets hold equally for the more general ANR-sets.\(^3\)

The Hilbert parallelopotope $Q$, that is, the product of the closed unit interval $[0, 1]$ with itself a countable number of times is a “universal” compact ANR in the sense that\(^4\) every compact ANR is homeomorphic to a neighborhood retract of $Q$. The classical theory of Borsuk makes good use of the imbedding of compact ANR-sets in $Q$. The problem solved here is that of finding a “universal” ANR.

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\(^3\) Ibid., pp. 272, 276, and 277, and Footnote 1, p. 279 and Footnote 3. Note that Theorem 12, Fundamenta Mathematicae, vol. 19 (1932), p. 229, is not true for general ANR-sets. In fact let $A = \sum S_n$ where $S_n$ is the plane circle of radius $2^{-n}$ and center $(3 \cdot 2^{-n}, 0)$; let $f(x, y) = (x, \ |y|)$ for $(x, y) \in A$ and let $f_n(x, y) = \begin{cases} (x, \ |y|), & \text{for} \ (x, y) \in A - S_n, \\ (x, y), & \text{for} \ (x, y) \in S_n. \end{cases}$ Then $f_n$—$f$ in $A$; $f$ can be extended to the half-plane $\{x > 0\}$, but none of the maps $f_n$ can. $A$ is an ANR-set. Theorem 16, Fundamenta Mathematicae, vol. 19 (1932), p. 230, is also false for general ANR-sets.

Strictly speaking, the problem as just stated has no solution; there is no single "universal" ANR, but rather a whole class of ANR-sets which together serve in the "universal" capacity. Such a class of ANR-sets is the collection of subsets of the Hilbert parallelepiped $Q \times [0, 1]$ which contains the open subset $Q \times (0, 1]$ of $Q \times [0, 1]$.

**Theorem 1.** For a separable metrizable space $X$ the following three conditions are equivalent:

1. $X$ is an ANR-set;
2. There is a homeomorphism $f$ of $X$ into $Q$ such that $f(X) \times [0]$ is a neighborhood retract of $f(X) \times [0] + Q \times (0, 1]$;
3. $f(X) \times [0]$ is a neighborhood retract of $f(X) \times [0] + Q \times (0, 1]$ for every homeomorphism $f$ of $X$ into $Q$.

$(1) \implies (3)$: If $f$ is a homeomorphism of an ANR-set $X$ into $Q$ then $f(X) \times [0]$ is an ANR-set. Since $Q$ is compact, so that $f(X) \times [0] \subset Q \times [0]$, it follows that $f(X) \times [0]$ is closed in $f(X) \times [0] + Q \times (0, 1]$. Hence $f(X) \times [0]$ is a neighborhood retract of $f(X) \times [0] + Q \times (0, 1]$.

$(3) \implies (2)$: Since $X$ is separable and metrizable a homeomorphism $f$ exists by Urysohn’s theorem.$^6$

$(2) \implies (1)$: Let $M$ be a separable metrizable space containing $X$ in which $X$ is closed and let $f$ be a homeomorphism of $X$ into $Q$. By Tietze’s theorem$^7$ there exists a continuous function $g$ defined on $M$ with values in $Q$ such that $g(x) = f(x)$ for every $x \in X$. Let $M$ be metrized, with metric $d$, and let $\rho(x) = \min \{1, d(x, X)\}$ for every $x \in M$. Let $h(x) = (g(x), \rho(x))$, so that $h$ is a continuous function defined on $M$ with values in $f(X) \times [0] + Q \times (0, 1]$ which has the property $h(M - X) \subset Q \times (0, 1]$. Let $V$ be a neighborhood of $f(X) \times [0]$ in $f(X) \times [0] + Q \times (0, 1]$ and let $U = h^{-1}(V)$ so that $U$ is a neighborhood of $X$ in $M$. If $r$ is a retraction of $V$ onto $f(X) \times [0]$ then the mapping$^8$ $f^{-1}\pi rh|U$, where $\pi$ denotes the projection of $Q \times [0]$ onto $Q$, is a retraction of $U$ onto $X$.

Kuratowski also gave an analogous generalization of the notion of absolute retract.$^2$ According to the extended definition a separable metrizable space is an absolute retract (AR) if it is a retract of every containing separable metrizable space in which it is closed.

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$^5$ The symbol $[0, 1]$ denotes the half-open interval $0 < t \leq 1$.


$^7$ Ibid., p. 73.

$^8$ If $B \subset B'$ and $e$ is a function defined on $B'$ then the notation $d = e|B$ means that $d$ is the function defined on $B$ such that $d(x) = e(x)$ for every $x \in B$. 
Theorem 1'. For a separable metrizable space $X$ the following three conditions are equivalent:

1. $X$ is an AR;
2. There is a homeomorphism $f$ of $X$ into $Q$ such that $f(X) \times [0]$ is a retract of $f(X) \times [0] + Q \times (0, 1]$;
3. $f(X) \times [0]$ is a retract of $f(X) \times [0] + Q \times (0, 1]$ for every homeomorphism $f$ of $X$ into $Q$.

The proof of this theorem is an obvious modification of the preceding proof.

Corollary. If $C$ denotes the open $n$-cell $0 < x_i < 1$ ($i = 1, \ldots, n$) and $D$ denotes the closed $n$-cell $0 \leq x_i \leq 1$ ($i = 1, \ldots, n$) then any set $E$ such that $C \subseteq E \subseteq D$ is an AR.

By condition (2') and a retraction of $Q \times [0, 1]$ onto $D \times [0, 1]$ it is sufficient to show that $E \times [0]$ is a retract of $E \times [0] + D \times (0, 1]$. This can be done by projecting from the point $(1/2, \ldots, 1/2, -1)$ of Euclidean $(n+1)$-space.

It may be worth noting that conditions (2) and (2') make possible a simpler proof of the Borsuk-Kuratowski theorem(s):

If $W$ is a closed subset of a normal space $Z$ and $X$ is an AR-set (ANR-set) then every continuous map of $W$ into $X$ can be extended to $Z$ (to a neighborhood of $W$ in $Z$).

In fact conditions (2) and (2') replace a theorem of Kuratowski which involves infinite polyhedra.

Theorem 2. An ANR is locally contractible. An AR is also contractible.

Using (2) we can suppose that our ANR-set $Y$ is contained in $Q \times [0]$ and that there is a retraction $r$ of an open neighborhood $V$ of $Y$ in $Y + Q \times (0, 1]$ onto $Y$. But $V$ is the intersection of $Y + Q \times (0, 1]$ with an open set $V'$ of $Q \times [0, 1]$. Let $y \in Y$ and let $S_\epsilon$ denote the $\epsilon$-sphere in $Q \times [0, 1]$ about the point $y$. Since $r$ is continuous there is a $\delta > 0$ such that the intersection $T_\delta$ of the $\delta$-sphere $S_\delta$ and $Y + Q \times (0, 1]$ is contained in $V'$, hence in $V$, and $r(T_\delta) \subseteq S_\epsilon$. Let $u_t$ denote a contraction of $S_\delta$ to a point $p \in S_\delta \cdot (Q \times (0, 1])$ which moves points rectilinearly, so that $u_t(x) \in Q \times (0, 1]$ for every $0 < t \leq 1$.

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11 But not uniformly. See the example in Footnote 3. This theorem was proved by Borsuk, Fundamenta Mathematicae, vol. 19 (1932), p. 237 for compact ANR-sets.
y ∈ Y · S₂. Then ru₃ Y · S₂ contracts Y · S₂ in Y · S₄. The second state-
ment is a consequence of Theorem 3'.

Theorem 3. A separable metrizable space X is an ANR if and only
if for every separable metrizable space M containing X (in which X need
not be closed!) there is a neighborhood U of X and a continuous function
h defined on X × [0] + U × (0, 1] with values in X such that h|X × [0, 1]
is a deformation.¹²

Suppose X is an ANR and M a separable metrizable space con-
taining X. We may assume that M ⊆ Q. By (2) and (3) there is an
open neighborhood V' of X × [0] + Q × (0, 1] and a retraction r of
V = V' · (X × [0] + Q × (0, 1]) onto X × [0]. Let λ(x) = d(x × [0],
Q × [0, 1] − V') for every x ∈ M and let U = π(V' · (Q × [0])) where,
as before, π denotes the projection of Q × [0] onto Q. Define for every
(x, t) ∈ X × [0] + U × (0, 1],

\[ h(x, t) = \begin{cases} 
\pi r(x, t), & \text{when } t \leq \lambda(x), \\
\pi r(x, \lambda(x)), & \text{when } t \geq \lambda(x). 
\end{cases} \]

Since λ is continuous and λ(x) > 0 when x ∈ U it follows that h is con-
tinuous.

Conversely, let U be a neighborhood of X in M = Q and let h be a
continuous function defined on X × [0] + U × (0, 1] with values in X
such that h(x, 0) = x for every x ∈ X. Then h is a retraction of
X × [0] + U × (0, 1] onto X × [0]. Furthermore X × [0] + U × (0, 1]
is a neighborhood of X × [0] in X × [0] + Q × (0, 1].

Theorem 3'. A separable metrizable space X is an AR if and only
if for any separable metrizable space M containing X there is a continu-
ous function h defined on X × [0] + M × (0, 1] with values in X such
that h|X × [0, 1] is a contraction.¹²

Let X be an AR and M a separable metrizable space containing X;
we may assume that M ⊆ Q. Let r be a retraction of X × [0] + Q × (0, 1]
onto X × [0]. Let p ∈ Q and let

\[ h(x, t) = \pi r(tp + (1 - t)x, t) \]

for every (x, t) ∈ X × [0] + M × (0, 1], where π is the projection of
Q × [0] onto Q. Then h maps X × [0] + M × (0, 1] continuously into X
and h|X × [0, 1] is a contraction of X.

The converse is proved as in Theorem 3.

¹² A deformation of X is a continuous mapping h of X × [0, 1] into X such that
h(x, 0) = x for every x ∈ X. If h(X, 1) is a point then h is called a contraction of X.
If $X$ is locally compact the deformation $h|X \times [0, 1]$ of Theorems 3 and 3' can be chosen in advance of $M$. For then there exists\footnote{Alexandroff and Hopf, *Topologie*, I, p. 93.} a compact set $M^*$ and a homeomorphism $g$ of $X$ into $M^*$ such that $M^* - g(X)$ is a point. (We can suppose $X$ not compact so that $M^* \neq g(X)$.) Let $M^* \subset Q$. The homeomorphism $g$ can be extended\footnote{Alexandroff and Hopf, *Topologie*, I, p. 93.} to a continuous mapping $g^*$ of $X$ into $M^*$ by defining $g^*(X - X) = M^* - g(X)$. The mapping $g^*$ of $X$ into $Q$ can be extended, by Tietze’s theorem, to a mapping $h$ of $M$ into $Q$. In the case of Theorem 3 let $h$ be the mapping of $X \times [0] + U \times (0, 1]$ into $X$ defined by

$$h(x, t) = g^{-1} \pi r(k(x), \min \{t, \lambda(x)\}),$$

where $U = g^{-1} \pi (V^* \cdot (Q \times [0]))$. In the case of Theorem 3' let $h$ be the mapping of $X \times [0] + M \times (0, 1]$ into $X$ defined by

$$h(x, t) = g^{-1} \pi r(tp + (1 - t)k(x), t).$$

In both cases $h|X \times [0, 1]$ is independent of $M$.

If $X$ is not locally compact it may not be possible to pick a deformation $h|X \times [0, 1]$ satisfying the conditions of Theorems 3 or 3' for all $M$. An example is the AR-set $\{0 \leq x \leq 1; y = 0\} + \sum_{n=1}^{\infty} \{x = 1/n; 0 \leq y \leq 1\}$.  

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