On nonacyclicity of the quotient space of $\mathbb{R}^3$
by the solenoid

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Abstract

It is well-known that the quotient space of the 3-dimensional Euclidean space $\mathbb{R}^3$ by the dyadic solenoid is not simply connected. We prove that the singular homology of this quotient space is uncountable.

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1. Introduction

Bing [1] was the first to observe that the quotient space $\mathbb{R}^3/\Sigma$ of the 3-dimensional Euclidean space $\mathbb{R}^3$ by the dyadic solenoid $\Sigma$ has a nontrivial fundamental group (a complete proof of this result was first published in [8,9]). However, not much is known about its properties. Therefore it is of interest to understand the nature of this group.

The quotient space $\mathbb{R}^3/\Sigma$ is homotopy equivalent to the dyadic projective telescope $\mathcal{P}_2T$. Bogley and Sieradski have shown that the fundamental group $\pi_1(\mathcal{P}_2T)$ is non-Abelian [2,11]. The purpose of the present paper is to show that the abelianization of the fundamental group $\pi_1(\mathbb{R}^3/\Sigma_\mathcal{P})$ of the quotient space $\mathbb{R}^3$ by any solenoid $\Sigma_\mathcal{P}$ is an uncountable group.

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Theorem 1.1. The quotient space \( \mathbb{R}^3 / \Sigma \) of \( \mathbb{R}^3 \) by any solenoid \( \Sigma \) is homotopy equivalent to the projective telescope \( \mathcal{P}T \) and the singular homology group \( H_1(\mathbb{R}^3 / \Sigma; \mathbb{Z}) \) is uncountable.

2. Preliminaries

Let \( S^1 \) be the oriented unit circle in the complex plane \( \mathbb{C} \). Consider the following inverse sequence \( \mathcal{P} \):

\[
P_0 \xleftarrow{f_0} P_1 \xleftarrow{f_1} P_2 \xleftarrow{f_1} \cdots
\]

where \( P_0 \) is a point, \( P_k \) is the circle \( S^1 \) and \( f_k : S^1 \to S^1 \) is the standard continuous mapping of degree \( n_k, n_k > 1, \) for every \( k > 0 \). The inverse limit \( \varprojlim \mathcal{P} \) is called the solenoid \( \Sigma \).

The space \( \Sigma \) is one-dimensional, compact and metric. It has a standard embedding into \( \mathbb{R}^3 \) (see, e.g., [5, pp. 230–231]). If \( n_k = 2 \) for all \( k \), then \( \Sigma \) is called the dyadic solenoid and denoted by \( \Sigma_2 \).

Let \( C(f_0, f_1, f_2, \ldots) \) be the infinite mapping cylinder (see, e.g., [6,7,10]) and let \( \widehat{\mathcal{P}} \) be its natural compactification by the solenoid \( \Sigma \). The projective telescope \( \mathcal{P}T \) is the one-point compactification of \( C(f_0, f_1, f_2, \ldots) \) by some point \{pt\}. We consider \{pt\} as the base point of \( \mathcal{P}T \) and the circles \( P_k \) for \( k = 1, 2, 3, \ldots \) as the natural subspaces of \( \mathcal{P}T \).

Hereafter, by homology we shall mean the singular homology with integer coefficients. Since the one-dimensional homology group of a path-connected space is the abelianization of the fundamental group, our results strengthen Bing’s theorem mentioned above [1,8,9].

To prove Theorem 1.1 we shall need the following results:

Theorem 2.1 (Borsuk [3,9]). Let \( W \) be a strong deformation retract of \( \widehat{W} \) and let \( X \) be any continuum in \( W \). Then \( W/X \) is a strong deformation retract of \( \widehat{W}/X \). Thus in particular, \( W/X \) and \( \widehat{W}/X \) have the same homotopy type.

Proposition 2.2. The compactum \( \mathcal{P}T \) is an absolute retract.

Proof. The proposition is a direct consequence of well-known results (see, e.g., [7, p. 104]). \( \square \)

Consider the following closed subset of \( S^1 \):

\[
A = \left\{ e^{2\pi i t} \in S^1 \mid t = \frac{1}{k}, \ k \in \mathbb{N} \right\}.
\]

The quotient space \( S^1/A \) is homeomorphic to the Hawaiian earring \( \mathcal{H} \), i.e., to the compact bouquet of a countable number of circles \( \{S^1_k\}_{k \in \mathbb{N}} \).

Let \( p : S^1 \to \mathcal{H} \) be the canonical projection, \( \mathbb{Z} \) the infinite cyclic group and \( \mathbb{Z}_n \) the finite cyclic subgroup of order \( n \) of \( S^1 \):

\[
\mathbb{Z}_n = \left\{ e^{2\pi i t} \in S^1 \mid t = \frac{k}{n}, \ k = 1, 2, \ldots, n \right\}.
\]
3. Proof of Theorem 1.1

Since the space \( \tilde{P} \) is a 2-dimensional compactum, it can be considered as a closed subspace of \( \mathbb{R}^3 \). Since \( \mathbb{R}^3 \) and (by Proposition 2.2) \( \tilde{P} \) is an absolute retract, \( \tilde{P} \) is a strong deformation retract of \( \mathbb{R}^3 \). The compactum \( \Sigma P \) is a subset of \( \tilde{P} \), therefore by Theorem 2.1 the quotient space \( \mathbb{R}^3 / \Sigma P \) is homotopy equivalent to the quotient space \( \tilde{P} / \Sigma P \), which is obviously homeomorphic to the projective telescope \( PT \).

Since the homotopy type of \( \mathbb{R}^3 / \Sigma P \) does not depend on the way in which \( \Sigma P \) is embedded into \( \mathbb{R}^3 \) (see Theorem 1 in [9]), we can assume that \( \Sigma P \) is embedded into \( \mathbb{R}^3 \) as the composition of the standard embeddings \( \Sigma P \subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^3 \times \mathbb{R}^2 \), where 0 is the origin of \( \mathbb{R}^2 \). By Theorem 2.1, \( \mathbb{R}^3 / \Sigma P \) is homotopy equivalent to \( \mathbb{R}^3 / \Sigma P \) and therefore to the projective telescope \( PT \). The first part of Theorem 1.1 is thus proved.

Suppose now that to the contrary, \( H_1(PT) \) were a countable group. Consider \( PT \) as the union: \( PT = C(f_0) \cup C(f_1, f_2, f_3, \ldots)^* \), where \( C(f_0) \) is the cylinder of the constant mapping \( f_0 : S^1 \rightarrow S^1 \) and therefore is a contractible space, and \( C(f_1, f_2, f_3, \ldots)^* \) is the one-point compactification of the infinite mapping cylinder \( C(f_1, f_2, f_3, \ldots) \). The intersection of these two subspaces of \( PT \) is the circle \( S^1 \). Thus it follows by the Mayer–Vietoris exact sequence:

\[
\rightarrow H_1(S^1) \rightarrow H_1(C(f_0)) \oplus H_1(C(f_1, f_2, f_3, \ldots)^*) \rightarrow H_1(PT) \rightarrow \cdots
\]

that the group

\[
H_1(C(f_1, f_2, f_3, \ldots)^*)
\]

is countable. (3.1)

Consider now \( C(f_{n+1}, f_{n+2}, f_{n+3}, \ldots)^* \) as a subspace of \( C(f_1, f_2, f_3, \ldots)^* \). Let \( X_n \) and \( p_n : C(f_1, f_2, f_3, \ldots)^* \rightarrow X_n \) be the corresponding quotient space and the quotient mapping. For every sequence of units and zeros \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) \), let \( g_\alpha : \mathcal{H} \rightarrow \mathcal{H} \) be the mapping such that

\[
g_\alpha|_{S^1} = \begin{cases} 
\text{the identity mapping onto its image}, & \text{if } \alpha_k = 1, \\
\text{the constant mapping into the base point}, & \text{if } \alpha_k = 0.
\end{cases}
\]

Let \( g \) be a mapping of \( \mathcal{H} \) to \( C(f_1, f_2, f_3, \ldots)^* \) which maps the base point of \( \mathcal{H} \) to the base point \( \{pt\} \) of \( C(f_1, f_2, f_3, \ldots)^* \) and such that the restriction \( g|_{S^1} \) only wraps once around the circle \( P_k \) in the positive direction.

The set \( \{g_\alpha\} \) is uncountable. However, the group \( H_1(C(f_1, f_2, f_3, \ldots)^*) \) is countable (3.1). Therefore there exist two sequences \( \alpha \) and \( \beta \) such that \( \alpha \neq \beta \) and such that for the mappings \( S^1 \xrightarrow{\beta} \mathcal{H} \xrightarrow{g_\alpha} \mathcal{H} \xrightarrow{g_\beta} C(f_1, f_2, f_3, \ldots)^* \) and \( S^1 \xrightarrow{\beta} \mathcal{H} \xrightarrow{g_\beta} C(f_1, f_2, f_3, \ldots)^* \) we obtain the same homomorphism of the corresponding homology groups:

\[
(g_\alpha \cdot g_\beta p)|_1 = (g_\alpha \cdot g_\beta p)|_1 : H_1(S^1) \rightarrow H_1(C(f_1, f_2, f_3, \ldots)^*).
\]

(3.2)

On the other hand, let \( m \) be the minimal number such that \( \alpha_m \neq \beta_m \). To the projection \( p_m : C(f_1, f_2, f_3, \ldots)^* \rightarrow X_m \) there correspond two homomorphisms of homology groups:

\[
H_1(S^1) \xrightarrow{p_m \beta} H_1(X_m) \text{ and } H_1(S^1) \xrightarrow{p_m \alpha \cdot g_\beta p} H_1(X_m).
\]

Since \( \alpha_k = \beta_k \) for \( k < m \) and \( \alpha_m \neq \beta_m \), by construction we have \( (p_m \cdot g_\alpha \cdot g_\beta p)|_1(1) \neq (p_m \cdot g_\beta p)|_1(1) \), contradicting (3.2).
Question 3.1. Let $X$ be the Case–Chamberlin continuum [4]. Is then the homology of quotient space $H_1(\mathbb{R}^3/X)$ nontrivial?

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