Homotopy properties of decomposition spaces

by

Neelima Shrikhande (Mt. Pleasant, Mich.)

Abstract. Let \( X \) be a continuum (compact, connected set) in \( E^n \). Then the homotopy type of the decomposition space \( E^n/X \) depends only on the shape of \( X \). We also show a necessary and sufficient condition for \( E^n/X \) to be locally simply connected. This is the "nearly-1-movable" property of continua described by D. R. McMillan. Thus the local simple connectedness of decomposition space also depends only on the shape of \( X \).

Introduction. Let \( X \) be a continuum (compact, connected set) in Euclidean \( n \)-space \( E^n \). We investigate the homotopy properties of decomposition space \( E^n/X \) obtained by identifying \( X \) to a point and giving the resulting space the quotient topology.

We first show that the homotopy type of \( E^n/X \) depends only on the shape of \( X \). This generalizes previous results of D. Henderson [7], S. Mardešić [9], and R. Geoghegan and R. Summerhill [6]. There are continua \( X \) and \( Y \) which have the same shape, but their decomposition spaces are not homeomorphic (for example, two arcs in \( E^2 \), one cellular and one noncellular). On the other hand, there are homeomorphic decomposition spaces of two continua \( X, Y \) where \( X, Y \) do not have the same shape.

D. R. McMillan [11] defined the concept of "nearly-1-movable". We show that the property of a continuum being nearly-1-movable is necessary and sufficient for \( E^n/X \) being locally simply connected. Thus by [12], this property is also equivalent to \( E^n/X \) being simply connected. As a corollary we get the results that \( E^2 \) modulo a solenoid or \( E^3 \) modulo the "Case-Chamberlin continuum" [4] are not simply connected. The first result was announced by R. H. Bing in [1]. The second result was shown by S. Armentrout. Both proofs are unpublished.

Throughout the paper we use the geometric approach to Shape theory as defined by Borsuk [2].

§ 1. We show that if two continua \( X \) and \( Y \) in \( E^n \) have the same shape then their decomposition spaces have the same homotopy type.

* The contents of this paper form part of the Author's Ph. D. thesis written at Madison, Wisconsin under the direction of Professor D. R. McMillan, Jr.
Let \( Q = \prod_{1}^{\infty} [-1, 1] \) denote the Hilbert cube. We identify \( E^n = \prod_{1}^{n} (-1, 1) \) in the first \( n \) factors of \( Q \), and the unit ball \( B^n = \prod_{1}^{n} [-\frac{1}{2}, \frac{1}{2}] \subset E^n \subset Q \).

If \( X \) is a continuum in \( B^n \), \( B^n \setminus X \) can be considered as a subset of \( E^n \setminus \mathbb{Q} \). We state a theorem of K. Borsuk [3] in this notation.

**Theorem (K. Borsuk).** Let \( W \) be a strong deformation retract of \( \mathbb{R} \). Let \( X \) be a continuum in \( W \). Then \( W \setminus X \) is a strong deformation retract of \( \mathbb{R} \setminus X \).

Thus in particular, \( \mathbb{R} \setminus X \) and \( \mathbb{R} \setminus \mathbb{Q} \) have the same homotopy type.

**Corollary.** Since \( B^n \) is a strong deformation retract of both \( Q \) and \( E^n \), therefore \( Q \setminus X \) and \( E^n \setminus X \) have the same homotopy type.

**Theorem 1.** Let \( X, Y \subset E^n \) be continua such that \( \text{Sh}(X) \subset \text{Sh}(Y) \). Then \( E^n \setminus X \) has the same homotopy type as \( E^n \setminus Y \).

**Proof.** Since we are considering \( E^n \) as embedded in the first \( n \) factors of \( Q \), \( X \) and \( Y \) are \( n \)-sets in \( Q \). Thus by Chapman [5],

\[ Q \setminus X \text{ is homeomorphic to } Q \setminus Y. \]

Let \( h: \mathbb{Q} \setminus X \to \mathbb{Q} \setminus Y \) be a homeomorphism.

Define \( \tilde{h}: Q \setminus X \to Q \setminus Y \) to be \( \tilde{h}(x) = h(x), x \notin X, \tilde{h}(x) = Y \). Then \( \tilde{h} \) is continuous since \( h \) is a proper map. Since \( h \) is a 1 to 1, continuous function between compact spaces, it is a homeomorphism. Thus \( Q \setminus X \cong Q \setminus Y \).

By corollary above \( E^n \setminus X \) has the same homotopy type as \( E^n \setminus Y \).

**Question.** Let \( X, Y \) be continua in \( E^n \). Let \( \text{Sh}(X) \supseteq \text{Sh}(Y) \). Does \( E^n \setminus X \) homotopically dominate \( E^n \setminus Y \)?

**Remark.** We know by [12] that if \( X, Y \) are continua in \( E^n \) (or \( \mathbb{Q} \)), \( \text{Sh}(X) \supseteq \text{Sh}(Y) \) and \( E^n \setminus X \) is simply connected, then \( E^n \setminus Y \) is also simply connected.

\( \mathbb{S} \). A compact set \( X \subset Q \) is said to be nearly-1-movable if for some (and hence for every) embedding of \( X \) in \( Q \), and each open set \( U \subset Q \) containing \( X \), there is an open set \( V \) containing \( X \) such that \( V \) nearly-1-moves towards \( X \) in \( U \).

That is, given any loop

\[ l: S^1 \to V, \]

and any open \( W \) containing \( X \), there is a map

\[ g: B^2 = \bigcup_{i=1}^{n} D_i \to U \]

\((D_i \text{ closed 2-cell } \subset \text{Int} B^2, i = 1, 2, \ldots, n, D_i \cap D_j = \emptyset, i \neq j \) such that

\[ g|_{\partial B^2} = l \quad \text{and} \quad g(\bigcup \partial D_i) \subset W. \]

In other words, every loop in \( V \) belongs to the normal closure in \( U \) of every neighborhood \( W \) of \( X \).

D.R. McMillan has shown [11] that 1-movability implies nearly-1-movability and that this implication is irreversible. The solenoids as also the ‘Case-Chamberlin continuum’ [4] are not nearly-1-movable.

We show first that nearly-1-movability is a shape property.

**Lemma 21.** Let \( X, Y \) be continua in \( Q \). If \( X \) is nearly-1-movable and \( \text{Sh}(X) \supseteq \text{Sh}(Y) \) then \( Y \) is nearly-1-movable.

**Proof.** There are fundamental sequences

\[ f = \{ f_i, X, Y \} \quad \text{and} \quad g = \{ g_i, X, Y \} \]

such that \( f \circ g = \text{id} \).

Let \( U \) be any open set containing \( Y \). Then there is

(i) \( U' \) containing \( Y \) and integer \( N \geq 1 \) such that

\[ f_i * g_i |_{U'} \subset f_{i+1} * g_{i+1} |_{U'} \subset \text{id} |_{U'} \]

for all \( k \geq N \).

(ii) There is a \( U' \) containing \( X \) and \( N' \geq 1 \) such that

\[ f_{i+1} |_{U'} \subset f_{i+1} |_{U'} \subset \text{id} |_{U'} \]

for all \( k \geq N' \).

(iii) There is a \( V \) containing \( Y \) and \( N' \geq 1 \) such that

\[ g_i |_{V} \subset g_i |_{V} \subset \text{id} |_{V} \]

for all \( k \geq N' \).

(iv) There is a \( V \) containing \( X \) and \( N' \geq 1 \) such that

\[ g_i |_{V} \subset g_i |_{V} \subset \text{id} |_{V} \]

for all \( k \geq N' \).

Thus it is easy to see that \( V \) nearly-1-moves towards \( Y \) in \( U \).

Thus nearly-1-movability is a shape property. To prove the if part of our main theorem we use the notion of local-1-connection, as defined by G. Kozlowski in [8].

**Definition.** The projection \( p: E^n \to E^n \setminus X \) is said to be a local-1-connection if for each open set \( U \) in \( E^n \setminus X \) containing \( X = p(X) \), there is an open \( V \subset E^n \setminus X \) such that every loop in \( p^{-1}(V) \) projects to a loop that is homotopic to a constant in \( U \).

**Theorem 2.** Let \( X \subset E^n \) be a continuum. Then \( X \) is nearly-1-movable if and only if \( E^n \setminus X \) is locally simply connected.

**Proof.** First we show that if \( X \) is nearly-1-movable then \( p: E^n \to E^n \setminus X \) is a local-1-connection.

Let \( U \) be an open set containing \( p(X) = X \). \( p^{-1}(U) \) is an open set in \( E^n \) and contains \( X \). Since \( X \) is nearly-1-movable, there is a sequence of open sets \( \{ V_i \}_{i=0}^{\infty} \) with the following properties

(i) \( V_0 = p^{-1}(U), X \subset V_i \) for \( i = 0, 1, 2, \ldots \),

(ii) \( V_{i+1} \subset V_i \),

(iii) every loop in \( V_i \) nearly-1-moves towards \( X \) in \( V_{i-1} \).
We let \( V = p(V_i) \), an open set. Let \( l: S^1 \to V \) be a loop. There is \( D_i: B^i - \bigcup\limits_{j \neq i} B^j \to U \) where each \( B^j \) is a 2-cell, \( B^j \cap B^i = \emptyset, \quad i \neq j \), \( \bigcup B^j \subset \text{int}(B^i) \) and \( \text{diam}(B^j) < 1 \), such that \( D_i \big|_{\partial B^j} = l, \quad D_i(B^j) = V_j, \quad i = 1, 2, \ldots, n. \)

Now \( D_i \big|_{\partial B^j} \) is a loop in \( V_j \) so there is \( D_i: B^j - \bigcup\limits_{j \neq i} B^j \to V_i \) such that diam \( B^j \leq \frac{1}{2} \) and \( D_i \big|_{\partial B^j} = D_i \big|_{\partial B^j} \), and \( D_i \big|_{\partial B^j} \) is a loop in \( V_j \).

We continue in this manner. Since the union of the \( i \)-th stage is contained in some 2-cell \( B_i^{-1} \) at the \((i-1)\)-st stage, it is possible to get a map \( D \) of \( B_i^{-1} \) minus a zero dimensional set \( S \). (This is possible since the diameter of each \( B^j \) is less than \( 1/2 \)).

We define a map from \( B^1 \) to \( U \) as follows:

\[
D^1(y) = \begin{cases} 
  p \circ D(y), & \text{if } y \in B^i - S, \\
  p(X), & \text{if } y \in S.
\end{cases}
\]

Since the image under \( D^1 \) of the union of \( \partial B^j \) at each stage is contained in \( V_{i+1} \), and the image of the zero dimensional set under \( D \) is contained in \( p(X) \), therefore \( D^1 \) is continuous. Thus \( D^1: B^i \to U \) extends \( p \circ l: S^1 \to p(V_i) = V \). Hence every loop in \( V_i \) projects to a loop which homotopes to a constant in \( U \). Thus \( p \) is a local-1-connection. To show that this implies that \( E^i/X \) is locally simply connected, we can apply lemma 1 of G. Kozlowski [8].

Conversely, assume \( E^i/X \) is locally simply connected. Let \( X \subset E^i, X \subset (0) \subset E^{i+1} \).

We consider \( X \) as a subset of \( E^i \times (0) \) embedded in \( E^{i+1} \) as shown above. We work in \( E^{i+1}/X \) to find sufficient space to shrink loops.

Consider this diagram:

\[
E^i \times E^1 \xrightarrow{p_{i+1}} E^{i+1}/X
\]

Define \( F: E^{i+1}/X \to E^i \times E^1 \) to be \( F(y_i) = (y_i, l). \) Then \( F \circ p_{i+1} = p_i \).

It is easy to show that \( F \) is well defined and continuous. Let \( U \) be an open set in \( E^{i+1}/X \) containing \( X = p_{i+1}(X) \).

\( F^{-1}(U) \) is an open set in \( E^{i+1} \) and contains \( X \). Let \( U^1 = F_{-1}(U) \cap E^1 \) which is open in \( E^1 \) and \( X \subset U^1 \).

Since \( E^i/X \) is locally simply connected, there is an open set \( V^1 \in E^i \times (0) \) with \( X \subset V^1 \subset U^1 \) such that every loop in \( F(V^1) \) shrinks in \( F(U^1) \).

There is an \( \varepsilon > 0 \) such that \( V^1 \times (-\varepsilon, \varepsilon) \subset F_{-1}(U) \). Let \( V = F_{-1}(V^1 \times (-\varepsilon, \varepsilon)) \); which is contained in \( U \). We want to show that each loop in \( V \) shrinks in \( U \). It is sufficient to show that \( p_{i+1} \) is a local 1-connection.

Let \( l: S^1 \to V^1 \times (-\varepsilon, \varepsilon) \). Then \( l \) is freely homotopic in \( V^1 \times (-\varepsilon, \varepsilon) \) to a loop \( l^1 \) in \( V^1 \times (0) \).

Now \( p_i(l(S^1)) = p_i(V^1 \times (0)) \subset E^i/X \times (0) \).

Hence \( p_i: S^1 \to p_i(V^1 \times (0)) \) extends to \( g: B^2 \to F(U^1) \), so \( F^{-1} \circ g \big|_{S^1} = p_{i+1} \big|_{S^1} \).

Thus \( p_{i+1} \circ l(S^1) \) shrinks in \( F(U) \subset E^{i+1}/X \). Therefore \( E^{i+1}/X \) is locally simply connected.

Now we show that \( X \) is nearly-1-movable as a subset of \( E^{i+1} \). Let \( U \) be open in \( E^{i+1} \) containing \( X \). Choose \( V^1 \subset p(U) \) by local simple connectedness. Let \( V = p^{-1}(V^1) \). Let \( l: S^1 \to V \) be a loop and let \( W, X \subset W \subset V, W \) open, be given.

We have to show that \( l \) belongs to the normal closure in \( U \) of \( W \).

We can assume that \( p \circ l(S^1) \) misses \( X \). For \( l \) is homotopic in \( V \) to a loop that misses \( X \). \( p \circ l: S^1 \rightarrow p(V) = V^1 \) extends to a map \( g: B^2 \to F(U) \).

Consider \( g^{-1}(P(X)) \) which is a compact set in the interior of \( B^1 \).

Then \( g^{-1}(P(X)) \subset g^{-1}(P(W)) \subset B^1 \).

We can find a finite number of disjoint simple closed curves \( R_1, R_2, \ldots, R_n \) with the following properties.

Let \( B_i \) denote the component of \( B^2 - R_i \) that misses \( B^1 \). Then the \( B_i \)'s are disjoint and \( \bigcup B_i \) contains \( g^{-1}(P(X)) \) and such that the images of these simple closed curves \( R_i \) lie in \( W \). (Such a collection of simple closed curves can be obtained by taking a brick decomposition of \( B^2 \) that has mesh smaller than \( \frac{1}{n} \text{dist}(g^{-1}(P(X)), B^2 - g^{-1}(P(W))) \))

and taking the relevant part of the boundary of the star of \( g^{-1}(P(X)) \).

Now \( g(B^2 - \bigcup\limits_{i=1}^{n} B_i) \) can be lifted to \( U \). Thus there is a map \( p^{-1} \circ g = \bar{g}: B^2 - \bigcup\limits_{i=1}^{n} B_i \to U \) such that \( \bar{g}(\partial B_i) \subset W, \quad \bar{g}_{|S^1} = l. \)

So \( l \) belongs to the normal closure in \( U \) of \( W \). Therefore \( X \) is nearly-1-movable.

§ 3. Movability properties are related to the UV properties [10] as follows. Property 1-UV for a compactum \( X \) clearly implies 1-movability.

Conversely,
THEOREM 3.1. Let $X$ be a continuum in $E^n$ having the property that for any neighborhood $U$ of $X$ the only loop that belongs to the normal closure in $U$ of each neighborhood $W$ of $X$ is the trivial loop. Then $X$ is nearly-$1$-movable if and only if $X$ is $1$-UV.

Proof. Let $X$ be nearly-$1$-movable. Let $U$ be an open set containing $X$. Choose $V$ so that each loop in $V$ belongs to the normal closure in $U$ of each open $W$. $X \subseteq W \subseteq V$. But only such loops are trivial loops. Thus $X$ is $1$-UV.

COROLLARY. If $X$ is as above, then $X$ has property $1$-UV if and only if $E^*X$ is locally simply connected.

Proof. Clear.

As a corollary, we get the following theorem of D. R. McMillan [10].

THEOREM. If $X$ is compact connected strongly $1$-acyclic, then $X$ is $1$-UV if and only if $E^*X$ is locally simply connected.

Proof. Strongly acyclic continua satisfy the property in Theorem 3.1.

References


Central Michigan University
Mt. Pleasant, Michigan

Accepté par la Rédaction le 16. 6. 1980

Yosida-Fukamiya’s theorem for $f$-rings

by Joaquin Pardo (Barcelona)

Abstract. We introduce the concept of super-infinitely small element and prove that in a commutative $f$-ring with unity the $J$-radical coincides with the set of all super-infinite small elements.

Preliminaries. We follow the notation and terminology of [1] and [5]. A lattice-ordered ring is an $f$-ring if $ax ≤ x ≤ ya$ whenever $x ≤ y$ and $a ≥ 0$. If we put $x^n = x^n ≤ 0$ and $x^n = (−x)^n ≤ 0$ and $|x| = x^n + x^−n$, then a lattice-ordered ring is a $d$-ring if $|xy| = |x| · |y|$, for all $x$, $y$. The term ideal must be understood in the ring-theoretic sense. A directed ideal is a $l$-ideal if $|y| ≤ |y|$, $y ≤ J ≤ x ≤ J$. We denote by $J$ the $l$-ideal generated by $a ∈ A$. Following [1], an element $a ∈ A$ such that $|a| = A$ is called a formal unity. An $l$-ideal $I$ is a band if, whenever a subset of $J$ has a supremum in $A$, that supremum belongs to $I$. The $J$-radical $J(A)$ of an $f$-ring $A$ is defined as the intersection of all maximal (two-sided) $I$-ideals, if there is any. Otherwise, $J(A) = A$ by definition. The ring $A$ is $J$-semisimple if $J(A) = 0$. An element $x ∈ A$ is infinitely small with respect to the element $y ∈ A$ whenever $n |x| ≤ |y|$ holds for $n = 1, 2, ...$. If we put $I_n(A) = \bigcup I_n(y)$, where $I_n(y) = \{x ∈ A \mid x$ is infinitely small with respect to $y\}$, then $A$ is Archimedean if and only if $I_n(A) = 0$. A lattice-ordered ring is Dedekind complete if every non-empty subset which is bounded above has a supremum.

Introduction. In vector lattices with a strong unit the Yosida–Fukamiya's theorem [7] asserts that the radical — intersection of all maximal $l$-vector subspaces — is the set of all infinitely small elements. Here, for a commutative $f$-ring with unity, we obtain a result that is parallel to that of Yosida–Fukamiya. But in this context infinitely small elements are no more appropriate and it has been necessary to introduce a notion of "smallness" related to the product of the ring: that of super-infinitely small element. And the set of all super-infinantly small elements of $A$ is proved to be $J(A)$.

Super-infinitely small elements and pseudoarchimedean rings.

Definition. The element $x$ of the lattice-ordered ring $A$ is called super-infinitely small element with respect to $y ∈ A$ whenever $|x| ≤ |x| ≤ |y|$ and $|x| ≤ |y|$ hold for every $a ∈ A$. 
