DISCRETENESS AND HOMOGENEITY OF THE TOPOLOGICAL FUNDAMENTAL GROUP

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Abstract. For a locally path connected topological space, the topological fundamental group is discrete if and only if the space is semilocally simply-connected. While functoriality of the topological fundamental group, with target the category of topological groups, remains an open question in general, the topological fundamental group is always a homogeneous space.

1. Introduction

The concept of a natural topology for the fundamental group appears to have originated with Witold Hurewicz [8] in 1935. It received further attention in 1950 by James Dugundji [2] and more recently by Daniel K. Biss [1], Paul Fabel [3], [4], [5], [6], and others. The purpose of this note is to prove the following folklore theorem.

Theorem 1.1. Let $X$ be a locally path connected topological space. The topological fundamental group $\pi_1^{\text{top}}(X)$ is discrete if and only if $X$ is semilocally simply-connected.

Theorem 5.1 of [1] is Theorem 1.1 without the hypothesis of local path connectedness. However, a counterexample of Fabel [6] shows that this stronger result is false. Fabel [6] also proves a weaker

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version of Theorem 1.1, assuming that $X$ is locally path connected and a metric space. In this note we remove the metric hypothesis.

Our proof proceeds from first topological principles, making no use of rigid covering fibrations [1] nor even of classical covering spaces. We make no use of the functoriality of the topological fundamental group, a property which was also a main result in [1, Corollary 3.4] but, in fact, is unproven [5, pp. 188–189]. Beware that the misstep in the proof of Proposition 3.1 in [1], namely the assumption that the product of quotient maps is a quotient map, is repeated in Theorem 2.1 of [7].

In general, the homeomorphism type of the topological fundamental group depends on a choice of basepoint. We say that $\pi_{1}^{\top}(X)$ is discrete, without reference to a basepoint, provided $\pi_{1}^{\top}(X,x)$ is discrete for each $x \in X$. If $x$ and $y$ are connected by a path in $X$, then $\pi_{1}^{\top}(X,x)$ and $\pi_{1}^{\top}(X,y)$ are homeomorphic. This fact was proved in Proposition 3.2 of [1], and a detailed proof is provided for completeness in section 4 of this paper. Theorem 1.1 now immediately implies the following.

**Corollary.** Let $X$ be a path connected and locally path connected topological space. The topological fundamental group $\pi_{1}^{\top}(X,x)$ is discrete for some $x \in X$ if and only if $X$ is semilocally simply-connected.

As mentioned above, it is open whether $\pi_{1}^{\top}$ is a functor from the category of pointed topological spaces to the category of topological groups. The unsettled question is whether multiplication

$$
\pi_{1}^{\top}(X,x) \times \pi_{1}^{\top}(X,x) \xrightarrow{\mu} \pi_{1}^{\top}(X,x)
$$

is continuous. By Theorem 1.1, if $X$ is locally path connected and semilocally simply-connected, then $\pi_{1}^{\top}(X,x)$, and, hence, the product $\pi_{1}^{\top}(X,x) \times \pi_{1}^{\top}(X,x)$ are discrete and so $\mu$ is trivially continuous. Continuity of $\mu$, in general, remains an interesting question.
Lemma 5.1 below shows that if $(X, x)$ is an arbitrary pointed topological space, then left and right multiplication by any fixed element in $\pi_1^{\text{top}}(X, x)$ are continuous self maps of $\pi_1^{\text{top}}(X, x)$. Therefore, $\pi_1^{\text{top}}(X, x)$ acts on itself by left and right translation as a group of self homeomorphisms. Clearly, these actions are transitive. Thus, we obtain the following result.

**Theorem 1.2.** Let $(X, x)$ be a pointed topological space. Then $\pi_1^{\text{top}}(X, x)$ is a homogeneous space.

This note is organized as follows. Section 2 contains definitions and conventions, section 3 proves two lemmas and Theorem 1.1, section 4 addresses change of basepoint, and section 5 shows left and right translation are homeomorphisms.

### 2. Definitions and Conventions

By convention, neighborhoods are open. Unless stated otherwise, homomorphisms are inclusion induced.

Let $X$ be a topological space and $x \in X$. A neighborhood $U$ of $x$ is *relatively inessential* (in $X$) provided $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. $X$ is *semilocally simply-connected* at $x$ provided there exists a relatively inessential neighborhood $U$ of $x$. $X$ is *semilocally simply-connected* provided it is so at each $x \in X$. A neighborhood $U$ of $x$ is *strongly relatively inessential* (in $X$) provided $\pi_1(U, y) \to \pi_1(X, y)$ is trivial for every $y \in U$.

The fundamental group is a functor from the category of pointed topological spaces to the category of groups. Consequently, if $A$ and $B$ are any subsets of $X$ such that $x \in A \subset B \subset X$ and $\pi_1(B, x) \to \pi_1(X, x)$ is trivial, then $\pi_1(A, x) \to \pi_1(X, x)$ is trivial as well. This observation justifies the convention that neighborhoods are open.

If $X$ is locally path connected and semilocally simply-connected, then each $x \in X$ has a path connected relatively inessential neighborhood $U$. Such a $U$ is necessarily a strongly relatively inessential neighborhood of $x$, as the reader may verify (see for instance, [9, Exercise 5, p. 330]).

Let $(X, x)$ be a pointed topological space and let $I = [0, 1] \subset \mathbb{R}$. The space

$$C_x(X) = \{ f : (I, \partial I) \to (X, x) \mid f \text{ is continuous} \}$$
is endowed with the compact-open topology. The function

\[ C_x(X) \xrightarrow{q} \pi_1(X, x) \]

\[ f \mapsto [f] \]

is surjective, so \( \pi_1(X, x) \) inherits the quotient topology, and one writes \( \pi_1^{\text{top}}(X, x) \) for the resulting topological fundamental group. Let \( e_x \in C_x(X) \) denote the constant map. If \( f \in C_x(X) \), then \( f^{-1} \) denotes the path defined by \( f^{-1}(t) = f(1-t) \).

3. Proof of Theorem 1.1

We prove two lemmas and then Theorem 1.1.

**Lemma 3.1.** Let \((X, x)\) be a pointed topological space. If \( \{[e_x]\} \) is open in \( \pi_1^{\text{top}}(X, x) \), then \( x \) has a relatively inessential neighborhood in \( X \).

**Proof:** The quotient map \( q \) is continuous and \( \{[e_x]\} \subset \pi_1^{\text{top}}(X, x) \) is open, so \( q^{-1}([e_x]) = [e_x] \) is open in \( C_x(X) \). Therefore, \( e_x \) has a basic open neighborhood

\[ e_x \in V = \bigcap_{n=1}^{N} V(K_n, U_n) \subset [e_x] \subset C_x(X), \]

where each \( K_n \subset I \) is compact, each \( U_n \subset X \) is open, and each \( V(K_n, U_n) \) is a subbasic open set for the compact-open topology on \( C_x(X) \). We will show that

\[ U = \bigcap_{n=1}^{N} U_n \]

is a relatively inessential neighborhood of \( x \) in \( X \). Clearly, \( U \) is open in \( X \) and, by (3.1), \( x \in U \). Finally, let \( f : (I, \partial I) \to (U, x) \). For each \( 1 \leq n \leq N \), we have

\[ f(K_n) \subset U \subset U_n. \]

Thus, \( f \in [e_x] \) by (3.1), so \([f] = [e_x]\) is trivial in \( \pi_1(X, x) \). \( \square \)

**Lemma 3.2.** Let \((X, x)\) be a pointed topological space and let \( f \in C_x(X) \). If \( X \) is locally path connected and semilocally simply-connected, then \( \{[f]\} \) is open in \( \pi_1^{\text{top}}(X, x) \).
Proof: As $q$ is a quotient map, we must show that $q^{-1}([f]) = [f]$ is open in $C_x(X)$. So let $g \in [f]$. For each $t \in I$, let $U_t$ be a path connected relatively inessential neighborhood of $g(t)$ in $X$. The sets $g^{-1}(U_t)$, where $t \in I$, form an open cover of $I$. Let $\lambda > 0$ be a Lebesgue number for this cover. Choose $N \in \mathbb{N}$ so that $1/N < \lambda$. For each $1 \leq n \leq N$, let

$$I_n = \left[ \frac{n - 1}{N}, \frac{n}{N} \right] \subset I.$$  

Reindex the $U_t$’s so that

$$g(I_n) \subset U_n \text{ for each } 1 \leq n \leq N.$$  

The $U_n$’s are not necessarily distinct, nor does the proof require this condition. For each $1 \leq n \leq N$, let $W_n$ denote the path component of $U_n \cap U_{n+1}$ containing $g(n/N)$, so

$$(3.2) \quad g\left(\frac{n}{N}\right) \in W_n \subset (U_n \cap U_{n+1}) \subset X.$$  

Consider the basic open set

$$(3.3) \quad V = \left( \bigcap_{n=1}^{N} V(I_n, U_n) \right) \cap \left( \bigcap_{n=1}^{N-1} V\left(\{\frac{n}{N}\}, W_n\right) \right) \subset C_x(X).$$  

By construction, $g \in V$. It remains to show that $V \subset [f]$. So, let $h \in V$. As $[g] = [f]$, it suffices to show that $[h] = [g]$.

By (3.3) we have

$$(3.4) \quad h\left(\frac{n}{N}\right) \in W_n \text{ for each } 1 \leq n \leq N - 1.$$  

For each $1 \leq n \leq N - 1$, let $\gamma_n : I \to W_n$ be a continuous path such that

$$\gamma_n(0) = h\left(\frac{n}{N}\right) \quad \text{and} \quad \gamma_n(1) = g\left(\frac{n}{N}\right).$$
which exists by (3.2) and (3.4). Let $\gamma_0 = e_x$ and $\gamma_N = e_x$. For each $1 \leq n \leq N$, define
\[
\begin{align*}
I & \overset{s_n}{\longrightarrow} I_n \\
t & \longmapsto \frac{1}{N} t + \frac{n-1}{N}
\end{align*}
\]
and let
\[
g_n = g \circ s_n \quad \text{and} \quad h_n = h \circ s_n.
\]

So, $g_n$ and $h_n$ are affine reparameterizations of $g|_{I_n}$ and $h|_{I_n}$, respectively. For each $1 \leq n \leq N$,
\[
\delta_n = g_n \ast \gamma_n^{-1} \ast h_n^{-1} \ast \gamma_n^{-1}
\]
is a loop in $U_n$ based at $g_n(0)$ (see Figure 1). As $U_n$ is a strongly rel-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{loop.png}
\caption{Loop $\delta_n = g_n \ast \gamma_n^{-1} \ast h_n^{-1} \ast \gamma_n^{-1}$ in $U_n$ based at $g_n(0)$.}
\end{figure}

\[
\pi_1(X, g_n(0)) = \langle \gamma_n^{-1} \rangle
\]

\[
[h] = [h_1 \ast h_2 \ast \cdots \ast h_N]
\]
\[
= \left[ \gamma_0^{-1} \ast h_1 \ast \gamma_1^{-1} \ast h_2 \ast \gamma_2 \ast \cdots \ast \gamma_{N-1}^{-1} \ast h_N \ast \gamma_N \right]
\]
\[
= [g_1 \ast g_2 \ast \cdots \ast g_N]
\]
\[
= [g],
\]
proving the lemma.

In the previous proof, the second collection of subbasic open sets in (3.3) is essential. Figure 2 shows two loops $g$ and $h$ based
at \( x \) in the annulus \( X = S^1 \times I \). All conditions in the proof are satisfied, except \( g(1/N) \) and \( h(1/N) \) fail to lie in the same connected component of \( U_1 \cap U_2 \). Clearly, \( g \) and \( h \) are not homotopic loops.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{Loops \( g \) and \( h \) based at \( x \) in the annulus \( X \).}
\end{figure}

**Proof of Theorem 1.1:** First, assume \( \pi_1^\text{top}(X) \) is discrete and let \( x \in X \). By definition, \( \pi_1^\text{top}(X,x) \) is discrete, so \( \{e_x\} \) is open in \( \pi_1^\text{top}(X,x) \). By Lemma 3.1, \( x \) has a relatively inessential neighborhood in \( X \). The choice of \( x \in X \) was arbitrary, so \( X \) is semilocally simply-connected.

Next, assume \( X \) is semilocally simply-connected and let \( x \in X \). Points in \( \pi_1^\text{top}(X,x) \) are open by Lemma 3.2, so \( \pi_1^\text{top}(X,x) \) is discrete. The choice of \( x \in X \) was arbitrary, so \( \pi_1^\text{top}(X) \) is discrete. \( \square \)

### 4. Basepoint change

**Lemma 4.1.** Let \( X \) be a topological space and \( x, y \in X \). If \( x \) and \( y \) lie in the same path component of \( X \), then \( \pi_1^\text{top}(X,x) \) and \( \pi_1^\text{top}(X,y) \) are homeomorphic.

**Proof:** Let \( \gamma : I \to X \) be a continuous path with \( \gamma(0) = y \) and \( \gamma(1) = x \). Define the function

\[
\begin{align*}
C_y(X) \xrightarrow{\Gamma} & C_x(X) \\
f \longmapsto & (\gamma^{-1} \ast f) \ast \gamma.
\end{align*}
\]
First, we show that $\Gamma$ is continuous. Let $I_1 = [0, 1/4]$, $I_2 = [1/4, 1/2]$, and $I_3 = [1/2, 1]$. Define the affine homeomorphisms

\[
\begin{align*}
I_1 & \xrightarrow{s_1} I & I_2 & \xrightarrow{s_2} I & I_3 & \xrightarrow{s_3} I \\
\text{and note that}
& \Gamma(f) \rightarrow X
& t \rightarrow \gamma^{-1} \circ s_1(t) & 0 \leq t \leq \frac{1}{4}
& t \rightarrow f \circ s_2(t) & \frac{1}{4} \leq t \leq \frac{1}{2}
& t \rightarrow \gamma \circ s_3(t) & \frac{1}{2} \leq t \leq 1.
\end{align*}
\]

Consider an arbitrary subbasic open set

\[
V = V(K, U) \subset C_x(X).
\]

Observe that $\Gamma(f) \in V$ if and only if

\[
\begin{align*}
\text{(4.1)} & \quad \gamma^{-1} \circ s_1(K \cap I_1) \subset U, \\
\text{(4.2)} & \quad f \circ s_2(K \cap I_2) \subset U, \quad \text{and}
\text{(4.3)} & \quad \gamma \circ s_3(K \cap I_3) \subset U.
\end{align*}
\]

Define the subbasic open set

\[
V' = V(s_2(K \cap I_2), U) \subset C_y(X).
\]

Observe that $f \in V'$ if and only if (4.2) holds. As conditions (4.1) and (4.3) are independent of $f$, either $\Gamma^{-1}(V) = \emptyset$ or $\Gamma^{-1}(V) = V'$. Thus, $\Gamma$ is continuous. Next, consider the diagram

\[
\begin{array}{c}
C_y(X) \xrightarrow{\Gamma} C_x(X) \\
\downarrow q_y \quad \downarrow q_x \\
\pi_1^{\text{top}}(X, y) \xrightarrow{\pi(\Gamma)} \pi_1^{\text{top}}(X, x).
\end{array}
\]

The composition $q_x \circ \Gamma$ is constant on each fiber of $q_y$, so there is a unique set function making the diagram commute, namely $\pi(\Gamma) : [f] \mapsto [\Gamma(f)]$. As $q_y$ is a quotient map, the universal property of quotient maps [9, Theorem 11.1, p. 139] implies that $\pi(\Gamma)$ is continuous. It is well known that $\pi(\Gamma)$ is a bijection [9, Theorem 2.1, p. 327]. Repeating the above argument with the roles of
$x$ and $y$ interchanged and the roles of $\gamma$ and $\gamma^{-1}$ interchanged, we see that $\pi(\Gamma)^{-1}$ is continuous. Thus, $\pi(\Gamma)$ is a homeomorphism as desired. □

5. Translation

Lemma 5.1. Let $(X, x)$ be a pointed topological space. If $[f] \in \pi_1^{\text{top}}(X, x)$, then left and right translation by $[f]$ are self homeomorphisms of $\pi_1^{\text{top}}(X, x)$.

Proof: Fix $[f] \in \pi_1^{\text{top}}(X, x)$ and consider left translation by $[f]$ on $\pi_1^{\text{top}}(X, x)$

$$\pi_1^{\text{top}}(X, x) \xrightarrow{L[f]} \pi_1^{\text{top}}(X, x)$$

Plainly, $L[f]$ is a bijection of sets. Consider the commutative diagram

(5.1)

$$\begin{array}{ccc}
C_x(X) & \xrightarrow{L_f} & C_x(X) \\
\downarrow{q} & & \downarrow{q} \\
\pi_1^{\text{top}}(X, x) & \xrightarrow{L[f]} & \pi_1^{\text{top}}(X, x),
\end{array}$$

where $L_f$ is defined by

$$C_x(X) \xrightarrow{L_f} C_x(X)$$

$$g \longmapsto f * g.$$  

First, we show $L_f$ is continuous. Let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. Define the affine homeomorphisms

$$I_1 \xrightarrow{s_1} I \hspace{1cm} I_2 \xrightarrow{s_2} I$$

$$t \longmapsto 2t \hspace{1cm} t \longmapsto 2t - 1$$

and note that

$$I \xrightarrow{f * g} X$$

$$t \longmapsto f \circ s_1(t) \hspace{1cm} 0 \leq t \leq \frac{1}{2}$$

$$t \longmapsto g \circ s_2(t) \hspace{1cm} \frac{1}{2} \leq t \leq 1.$$
Consider an arbitrary subbasic open set
\[ V = V(K, U) \subset C_x(X). \]
Observe that \( f * g \in V \) if and only if
\begin{align}
(5.2) & \quad f \circ s_1(K \cap I_1) \subset U \\
(5.3) & \quad g \circ s_2(K \cap I_2) \subset U.
\end{align}
Define the subbasic open set
\[ V' = V(s_2(K \cap I_2), U) \subset C_x(X). \]
Observe that \( g \in V' \) if and only if (5.3) holds. As condition (5.2) is independent of \( g \), either \( L_f^{-1}(V) = \emptyset \) or \( L_f^{-1}(V) = V' \). Thus, \( L_f \) is continuous. The composition \( q \circ L_f \) is constant on each fiber of the quotient map \( q \) and (5.1) commutes, so the universal property of quotient maps [9, Theorem 11.1, p. 139] implies that \( L[q] \) is continuous.

Applying the previous argument to \( f^{-1} \), we get \( L_f^{-1} = L[f^{-1}] \) is continuous and \( L[f] \) is a homeomorphism. The proof for right translation is almost identical. \( \square \)

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