ARTIN PRESENTATIONS OF COMPLEX SURFACES

J. S. CALCUT AND H. E. WINKELNKEMPER

Abstract. We construct Artin presentations of infinitely many complex surfaces. Namely, for all elliptic surfaces $E(n)$, in particular for the Kummer surface $K_3$. Thus, not only does AP theory contain an analogue of Donaldson’s Theorem, but also a purely group-theoretic theory of Donaldson and Seiberg-Witten invariants.

Not surprisingly, our explicit Artin presentations for the Kummer surface are approachable with a computer using, say, MAGMA and provide a plethora of interesting examples pertaining to knot theory in $\mathbb{Z}$-homology 3-spheres.

1. Introduction

In the purely group-theoretic theory of Artin presentations, a smooth, compact, connected, simply-connected 4-manifold $W^4(r)$ with a connected boundary $\partial W^4(r) = M^3(r)$ is already determined, and can be reconstituted, from a certain presentation (an Artin presentation) of the fundamental group of its boundary $[W1]$. If the boundary is $S^3$ then of course the Artin presentation presents the trivial group. Even in this case the Artin presentation already encodes all of the smooth structure of the 4-manifold. Thus, it makes sense to ask whether an arbitrary, smooth, closed, connected, simply-connected 4-manifold is given by an Artin presentation.

We extend important work of Harer, Kas and Kirby [HKK] and show that all elliptic surfaces $E(n)$ admit Artin presentations. This gives the first bridge between AP theory and algebraic geometry. These Artin presentations are of special interest due to the fact that complex algebraic surfaces possess non-trivial Donaldson invariants. In particular, this augments the remarkable fact (Theorem 1 of [W1], [R] p. 621) that Donaldson’s Theorem, despite being proved with gauge theory/connections (i.e. the smooth continuum), persists and survives the radical, discrete, purely group theoretic holography of AP theory.

The following illustrates the AP theory program concerning the computation of Seiberg-Witten and Donaldson invariants and shows that the group theoretic AP encoding goes much deeper than e.g. the mere encoding of a group through its presentation:

Recall González-Acuña’s formula, [CS] p. 66, for the Rohlin invariant of a $\mathbb{Z}$-homology 3-sphere $\Sigma^3(r)$ given by an Artin presentation $r \in \mathcal{R}_n$ (for clarity we consider here only the case where $A(r)$ is the identity matrix, see section 2.1 for notation):

$$\mu(\Sigma^3(r)) = \frac{d^2 - 1}{8} \mod 2,$$

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where \( d = \triangle(-1) \), \( \triangle \) being the Alexander polynomial of the associated presentation:

\[
\langle x_1, \ldots, x_n \mid x_1 r_1 = r_1 x_2, x_2 r_2 = r_2 x_3, \ldots, x_{n-1} r_{n-1} = r_{n-1} x_n \rangle,
\]

where the group obviously abelianizes to \( \mathbb{Z} \).

This remarkable formula is entirely from the discrete theory of finitely presented groups: there is no need to mention cobordisms, spin structures, skein methods, Heegaard decompositions, representations into \( SU(2) \), Riemannian metrics, infinite dimensional or moduli spaces, or indeed even the smooth continuum, nor do any metric dependence, wall crossing, or word problems arise here.

We remark that González-Acuña’s formula already shows that an analogue of Floer theory should also appear in AP theory since the Rohlin invariant is the Euler characteristic (mod 2) in Floer theory. In fact, we suspect that ‘the 8 of González-Acuña is the 8 of Floer’.

Concerning the importance of relating Donaldson and Floer theory, both mathematically and physically, see [D] p.63 and [Wi1] p.352.

Consider the more general problem concerning the relative Donaldson invariants [TB],[Wi1] of \( W^4(r) \) which, when \( A(r) \) is unimodular, take values in the Floer homology of \( \partial W^4(r) = \Sigma^3(r) \).

The computational program of AP theory can be stated as: these invariants and others should be computed solely in function of the Artin presentation \( r \) in the discrete theory of finitely presented groups, just as, with González-Acuña’s formula, this was done for the Rohlin invariant of \( \Sigma^3(r) \).

This is entirely in the purely group-theoretic spirit of the Princeton School of Artin, Fox, Lyndon, Papakyriakopoulos, Stallings, et al. and extends their approach, as far as 3D/4D manifold theory is concerned, to its natural metamathematical boundary.

Immediate natural, important general questions arise (both mathematical and physical):

1. Since AP theory dispenses not only with metrics but even topology, what becomes of Witten’s celebrated Feynmanian formulation of Donaldson’s invariants as correlation functions/expectation values [D] p. 53, [Wi2], [Wi3], [AJ], [Di] pp. 36, 39? What is the topologically independent (i.e. purely AP theoretical) analogue of Witten’s metric independent Lagrangian for the Casson theory [AJ] p.121? What does González-Acuña’s formula for the Rohlin invariant suggest? Is the mysterious question about the relationship between the Donaldson invariants of oppositely oriented \( X^4 \) related to the purely group-theoretic one of finding the inverse in \( \mathbb{R}_n \) of an Artin presentation?

2. In the absence of moduli spaces, etc., is Witten’s “mass-gap” discussion regarding Donaldson theory, [Wi3] pp.289-291, still relevant in AP theory?

3. Is the Denjoy-like inequivalence between Seiberg-Witten theory and Donaldson theory detectable in AP theory? Recall that Seiberg-Witten theory requires spinors and the Dirac operator, i.e. an underlying \( C^1 \) structure, whereas Donaldson’s theory is valid on the wider class of Lipschitz manifolds [D] p.69, [S], [DS].
4. In general, the word problem obstructs the study of arbitrary smooth 4-manifolds. Although 4-manifolds in AP Theory are simply connected, we can still ask whether the group-theoretical physical questions of Geroch-Hartle \([GH]\) (see also \([F]\)) are still relevant when transferred to the group theory of 3-manifolds. Theorem I of \([W1]\) seems to illustrate a purely group-theoretic Bohm-Aharanou phenomenon.

5. AP Theory does not just dispense with the smooth continuum, but also dispenses with integer (co)homology/intersection theory since all of this information is already given simply by the symmetric integer matrix \(A(r)\). Hence, should e.g. the Kronheimer-Mrowka canonical basic class of \(W^4(r)\), when \(\partial W^4(r) = S^3\), \([D]\) p.52, \([K]\), \([St]\), be already determined with Number Theory, à la Elkies \([E]\), \([D]\) p.67 and Borcherds \([B]\), thus explaining the persistence of invariants constructed with the aid of a complex structure when this structure does not exist? For the same reason, difficult ‘minimal genus’ and ‘simple type’ problems, \([D]\) p.68, \([St]\) p.156, should be studied in this, their ultimate natural context, where artificial complications caused by the use of the smooth continuum are absent.

It does not seem surprising, due to the basic nature of the \(K3\) complex surface (e.g. it is the only 4D, closed, simply connected Calabi-Yau manifold and its quadratic form is the first even non-Donaldson form), that our Artin presentations lead to several interesting and instructive examples (section 3 ahead) which complement and extend to the ‘softer’ non-Donaldson case those examples obtained from such matrices as \(E_8\), \(\phi_{4n}\), and the Coxeter-Todd extremal duodenary matrix \(2D^{12}_2\) \([W1]\).

2. The Artin presentations

The purpose of this section is to construct Artin presentations for all elliptic surfaces \(E(n)\). This is carried out completely for \(E(2)\), which is diffeomorphic to the Kummer surface \(K3\) \([GS]\), p.74, and follows mutatis mutandis for the others. The organization runs as follows: 2.1 is a brief discussion of Artin presentations and framed pure braids, in 2.2 we obtain a surgery diagram for \(E(n)\) that is a framed pure braid, 2.3 provides an explicit algorithm (fixing all conventions) for obtaining an Artin presentation from a framed pure braid, and 2.4 combines everything obtaining the desired Artin presentation for \(K3\).

(2.1) Artin presentations and pure braids. We begin by reviewing some of the fundamentals of AP theory. For a rigorous introduction to AP theory, proofs of the statements made below and a thorough bibliography we refer the reader to \([W1]\).

Let \(F_n = \langle x_1, \ldots, x_n \rangle\) be the free group on \(n\)-generators. An Artin presentation \(r\) is a balanced presentation \(r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle\) satisfying the equation:

\[(AC) \quad x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n),\]

in \(F_n\), which we will refer to as the Artin condition. The set of all Artin presentations on \(n\)-generators is denoted \(\mathcal{F}_n\) and forms a group. By \(\Omega_n\) we mean the compact 2-disk with \(n\)-holes and boundary \(\partial \Omega_n\) equal to the disjoint union
of $\partial_0, \partial_1, \ldots, \partial_n$ (see [W1] p.225). An Artin presentation $r \in \mathcal{R}_n$ determines, among other things, the following:

- $\pi(r)$: the group presented by $r$,
- $M^3(r)$: a closed orientable 3-manifold,
- $W^4(r)$: a smooth compact connected simply-connected 4-manifold,
- $A(r)$: an $n \times n$ symmetric integer matrix,
- $h(r)$: a self diffeomorphism of $\Omega_n$ unique up to isotopy fixing $\partial \Omega_n$ with $h|_{\partial \Omega_n} = id$.

The relationships between these objects are canonical. The manifold $M^3(r)$ bounds $W^4(r)$, has fundamental group isomorphic to $\pi(r)$, and is the open book defined by $h(r)$. The symmetric matrix $A(r)$ is the exponent sum matrix of $r$ and also represents the intersection form of $W^4(r)$. The manifold $M^3(r)$ is a $\mathbb{Z}$-homology 3-sphere if and only if $\det A(r) = \pm 1$, and in this case we write $\Sigma^3(r)$ instead of $M^3(r)$.

An Artin presentation $r \in \mathcal{R}_n$ also determines an automorphism of $F_n$ by the mapping $x_i \mapsto r_i x_i r_i^{-1}$. Namely, this is the automorphism $h_0 : \pi_1(\Omega_n, p_0) \to \pi_1(\Omega_n, p_0)$ where $p_0$ is a distinguished point in $\partial_0 \subset \partial \Omega_n$ and $x_1, \ldots, x_n$ represent the canonical generators (see Figure 9 ahead and [W1] p.225 and p.244). This view will prove useful when composing Artin presentations.

As pointed out in [W1], $\mathcal{R}_n$ is canonically isomorphic to $P_n \times \mathbb{Z}^n$, the framed pure braid group, where $P_n$ is the pure braid group on $n$-strands. To see this, notice that $r \in \mathcal{R}_n$ determines $h = h(r)$ and $h$ can be realized concretely in $\mathbb{R}^3$ by taking $\Omega_n \times I$ (where $I$ denotes the closed unit interval), suitably braiding the inner boundary tubes with one another, and twisting the inner boundary tubes by some integer numbers of complete revolutions (see [W1] p.245). Twisting the inner tubes can be accomplished by elementary Dehn twists about the $\partial_i$ and these Dehn twists commute with all others. This braiding/twisting of the inner boundary tubes is easily seen to be equivalent to specifying both a pure braid (pure as $h|_{\partial \Omega_n} = id$) and an integer (the ‘framing coefficient’) for each strand.

Let $r \in \mathcal{R}_n$. The manifold $W^4(r)$ is defined in [W1] p. 250 as follows. Embed $\Omega_n$ in $S^2$ and extend $h$ to all of $S^2$ by the identity. Then, extend this map to a self diffeomorphism of all of $D^3$, calling the result $H = H(r)$ (which is unique up to isotopy). Letting $W(H)$ be the mapping torus of $H$, $W^4(r)$ is defined to be $W(H)$ union $(n + 1)$ 2-handles attached canonically. Notice that $W(H)$ is diffeomorphic to $D^3 \times S^1 (= 0$-handle $\cup 1$-handle) as all orientation preserving self diffeomorphisms of $D^3$ are smoothly isotopic to the identity. We wish to examine this construction more closely. The self diffeomorphism $h$ of $\Omega_n$ can be realized, as described in the previous paragraph, in $\mathbb{R}^3$ as $\Omega_n \times I$ with the inner boundary tubes braided and twisted; the map $h$ of $\Omega_n$ is then obtained by bending the twisted $\Omega_n \times I$ around and sticking the ends $\Omega_n \times 0$ and $\Omega_n \times 1$ together in the canonical way, exactly as one does to close a braid. To construct $H$, one can first extend $h$ to $D^2$ by taking the twisted $\Omega_n \times I$ and filling in the $n$
inner boundary tubes with \(n\) copies of \(D^2 \times I\). One must take some care here. For each boundary tube \(\partial_i \times I, i = 1, \ldots, n\), let \(p_i\) be a distinguished point (see Figure 9 ahead and [W1] p.225). Let \(\ast\) be a distinguished point in \(\partial D^2\). Then, when filling the \(i\)th boundary tube \(\partial_i \times I\) with \(D^2 \times I\) one must attach \(\ast \times I\) to \(p_i \times I\) and fill with the identity at the ends \(\partial_i \times 0\) and \(\partial_i \times 1\). Now, \(h\) has been extended to \(D^2\) and is concretely realized as \(D^2 \times I\) by sticking the ends together as when closing a braid; call this intermittent mapping torus \(M(h)\) which is diffeomorphic to \(D^2 \times S^1\). Now, extending the map to \(D^3\) is trivial (again, \(h|_{\partial D^n} = id\)) and one immediately sees that the 2-handle attached corresponding to \(\partial_0\) cancels the 1-handle from the open book construction. Moreover, this cancellation occurs without disturbing the rest of the boundary of \(W(H)\). Thus, we are left with a 0-handle (i.e. \(D^4\)) with boundary \(S^3\) containing a very nice copy of \(M(h)\). To obtain \(W^4(r)\) we now attach the remaining \(n\) 2-handles to \(D^4\) along the copies of \(D^2 \times S^1\) in \(M(h)\) in the canonical way.

Summarizing the previous two paragraphs, an Artin presentation \(r\) determines a framed pure braid \(\beta\) in \(\mathbb{R}^3\) (which is the same as in \(S^3\)) and \(W^4(r)\) is obtained from \(D^4\) by attaching 2-handles according to \(\beta\). In the language of the Kirby calculus, all \(W^4(r)\)s are ‘2-handlebodies’ ([GS], p.124). For more on the manifolds \(W^4(r)\) see section 4.

**Remark (2.1.1).** One subtle but important distinction that must be made here between an \(r \in \mathcal{R}_n\) and a framed pure braid in \(S^3 = \partial D^4\) is that in an Artin presentation the framings are canonically included (they are not ‘put in by hand’ as in the Kirby calculus) thus, e.g. avoiding serious self-linking problems [Wi1], p.363. In fact, a moment of reflection by the reader should reveal that without this ‘canonicity’ one would not obtain the purely group theoretic analogue of Donaldson’s theorem [W1], p.240 Theorem 1, and its important consequences. See also [W1], p.241 and [W3].

Hence, one task to obtain an Artin presentation for a specific 4-manifold is to obtain a surgery diagram for the manifold that is a framed pure braid in \(S^3\) and then determine the corresponding Artin presentation from this framed pure braid. Of course, saying an Artin presentation \(r\) gives a closed 4-manifold \(X^4\) means that \(M^3(r) = S^3\) and \(W^4(r) \cup D^4 = X^4\) (i.e. close up with a 4-handle). We pursue this task in sections 2.2-2.4 below. We abuse notation and say an Artin presentation or a surgery diagram gives a closed 4–manifold when it actually presents the closed manifold minus the interior of a 4-handle (which can only be attached in one way, so there is no ambiguity).

We close this section by recalling useful knot theoretic structures in AP Theory. The simplicity of these structures allows us to avoid doing surgery ‘by hand’, avoids self-linking problems, etc. by use of a computer algebra system such as MAGMA and significantly adds to the power of AP Theory. We point out that, as usual, everything is group theoretic.

Fix \(r \in \mathcal{R}_n, r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle\), with \(\det A(r) = \pm 1\), in particular \(\Sigma^3(r)\) is a \(\mathbb{Z}\)-homology 3-sphere. There are \(n + 1\) distinguished knots in \(\Sigma^3(r)\) that are defined by the boundary circles \(\partial_0, \ldots, \partial_n\) of \(\Omega_n\) and we denote these knots by \(k_0, \ldots, k_n\). Let \(c_i\) denote the complement of \(k_i\) in \(\Sigma^3(r)\) and let \(G_i\) denote the fundamental group of \(c_i\). Since \(A(r)\) is unimodular, \(A(r)^{-1}\) is also a symmetric integer matrix and, in fact, is the linking matrix of the knots \(k_i\),
We let $b_{ij}$ denote the $ij$th entry of $A(r) - 1$ (abbreviating $b_{ii}$ to just $b_i$) and let $s = \sum_{ij} b_{ij}$. In $\Sigma^3(r)$, the self linking number of $k_0$ is $s$ and of $k_i$, $i \neq 0$, is $b_i$. We let $m_i$, $l_i$ denote the peripheral structure of the knot $k_i$, which consists of two special commuting elements in $G_i$, where $m_i$ is a meridian of $k_i$ and $l_i$ is homologically trivial in the complement of $k_i$. Then, we have:

$$G_0 = \langle x_1, \ldots, x_n \mid r_1 = r_2 = \cdots = r_n \rangle,$$

$$m_0 = \text{any } r_i,$n

$$l_0 = x_1 x_2 \cdots x_n m_0^{-s},$$

and for $i = 1, \ldots, n$ we have:

$$G_i = \langle x_1, \ldots, x_n \mid r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n \rangle,$$

$$m_i = r_i,$n

$$l_i = x_i m_i^{-b_i}.$$

Two remarks are in order. First of all, we get all knots and links in any arbitrary closed, orientable 3-manifold this way (González-Acuña unpublished). Second, the definition given here of $G_i$ for $i \neq 0$ appears to be slightly different from that given in [W1], p.227, but in fact the two are equivalent (this was pointed out to the second author by González-Acuña). This follows since the Artin Condition (AC) implies that in $G_i$ (definition given here) we have:

$$x_1 x_2 \cdots x_n = x_1 x_2 \cdots x_{i-1} (r_i^{-1} x_i r_i) x_{i+1} \cdots x_n,$$

which immediately implies that $x_i = r_i^{-1} x_i r_i$ in $G_i$. That is, $(x_i, r_i) = 1$ in $G_i$ (where $(a, b)$ is MAGMA notation for the commutator $a^{-1} b^{-1} a b$), showing the two definitions are equivalent. In fact, for $i \neq 0$, $m_i$ and $l_i$ commuting in $G_i$ is equivalent to $x_i$ and $r_i$ commuting in $G_i$.

**Pure braid for $E(n)$**. Our starting point is the framed link diagram in [HKK], p.66 (see also [GS], p.305) that presents a 2-handlebody with boundary $S^3$ and gives $E(n)$ upon closing up with a 4-handle. (As mentioned earlier, we abuse notation and say this diagram presents $E(n)$ where no confusion should arise.) By straightforward isotopy of the outer strand (the trefoil) we obtain Figure 1. The two large bands both represent $6n - 2$ strands, each strand with framing $-2$. A box containing ‘$-1$’ represents a twist of all strands (as when twisting ribbon) in the direction corresponding to a negative crossing in our orientation convention in Figure 8. We refer to the trefoil in Figure 1 as T and to the small circle linking it as S, which have framings 0 and $-n$ respectively.

All circles formed by closing a pure braid are individually not knotted, so the first step is to unknot the trefoil $T$. To accomplish this, one performs a 2-handle slide on $T$; in practice this corresponds to performing a band sum of $T$ with a parallel curve to another knot $K$ representing the framing on $K$ (see [GS], pp.141-143). Here we slide $T$ over the innermost circle in the left large band using the trivial band as in Figure 2. One checks that the curve in Figure 2 that $T$ is being band summed with is a parallel curve to the innermost strand and has linking number $-2$ with it (don’t forget the ‘$-1$’ box!). Let $T'$ denote the result of 2-handle sliding $T$. Figure 3 is obtained from Figure 2 by isotopy, in particular grab the part of $T'$ in Figure 2 that hangs below the two
Figure 1. Surgery diagram for $E(n)$. The large bands represent $6n - 2$ strands and all framings equal $-2$, except the trefoil $T'$ with framing 0 and the small circle $S$ linking it with framing $-n$.

large bands and swing it back and then up (other minor changes by isotopy here should be obvious). Straightforward isotopy of Figure 3 produces Figure 4 where it is apparent that $T'$ is not knotted.

It does not seem possible to isotop Figure 4 to a pure braid, so we perform another 2-handle slide. This time, slide $T'$ over the outermost strand in the right large band (again using a trivial band to band sum with) as shown in Figure 5. After a little isotopy one obtains Figure 6 (ignoring the hatched rectangle for the moment). Let $T''$ denote the result in Figure 6 of sliding $T'$ ($S$ is unchanged).

Now, Figure 6 isotops nicely to a pure braid. To see this, take the hatched rectangle in Figure 6, grab its upper left long boundary edge and pull it around, making a rather large (ambient) expansion of the hatched rectangle into a large backwards ‘C’ shape (the short dimension of the hatched rectangle extends

Figure 2. A 2-handle slide of $T$ over the innermost curve in the left large band using the indicated parallel curve and dashed band.
and bends around). Except for $S$, one now has a pure braid. A little more straightforward isotopy produces Figure 7, which is a pure braid for $E(n)$. The hatched rectangle does not appear in Figure 7, but one imagines it bending around on the right-hand side to close the braid. Figure 7 contains a total of $12n - 2$ strands: the two large bands each represent $6n - 2$ strands (each strand therein has framing $-2$), the $(12n - 3)^{rd}$ strand (second from the right) is $T''$, and the $(12n - 2)^{nd}$ strand (right-most) is $S$ with framing $-n$.

It remains to determine the framing on $T''$ (this is the only one that changed), which is calculated using the formula in [GS] p.142. The first 2-handle slide results in $T'$ with framing $-2$ since the relevant (signed, according to handle addition or subtraction) linking number is 0. The second 2-handle slide results in $T'''$ with framing still $-2$ since in this case the relevant signed linking number (whose overall sign is independent of orientation choices) is equal to

![Figure 3](image1.png)

**Figure 3.** The result $T'$ of 2-handle sliding $T$.

![Figure 4](image2.png)

**Figure 4.** The result of isotoping $T'$ (and $S$), which is not knotted.
Figure 5. A 2-handle slide of $T'$ over the outermost curve in the right large band using the indicated parallel curve and dashed band.

+1 implying $\pm 2lk(\cdot, \cdot) = +2$. Thus, in the pure braid diagram for $E(n)$ in Figure 7 all framings equal $-2$ except for the right-most strand which has framing $-n$. In particular, for the Kummer surface $E(2)$ all framings equal $-2$.

Remark (2.2.1). In Figure 1, the two large bands together form the compactified Milnor fiber $M_c(2, 3, 6n - 1)$ with boundary the Seifert fibered $\mathbb{Z}$-homology 3–sphere $\Sigma(2, 3, 6n - 1)$ and the trefoil union the small circle linking it form the Gompf nucleus $N(n)$ (see [GS], sec. 3.1, 6.3, 7.3 and 8.3). It is clear from the above that all Milnor fibers $M_c(2, 3, 6n - 1)$ admit Artin presentations.

(2.3) An Algorithm. Given a framed pure braid in $\mathbb{R}^3$, we wish to construct the corresponding Artin presentation. To make this explicit, we must fix some conventions. We will use $\beta$ to denote both a braid and a framed braid, where no confusion should arise. As usual, braids will be drawn as generic diagrams.

Figure 6. The result $T''$ of 2-handle sliding $T$. The hatched rectangle will be used to isotop to a pure braid.
in the plane with the strands ordered 1, 2, ..., n from left to right. We read our braids upwards, especially when composing them. In particular, each strand is oriented up. For a pure braid $\beta$, $C_i$ will denote the oriented circle consisting of the $i^{th}$ strand and the trivial segment that would close that strand upon closing the braid (the orientation is inherited from that of the corresponding braid strand). Crossings in any oriented generic link diagram in the plane are assigned a sign as in Figure 8. If $C_1$ and $C_2$ are two oriented circles in a generic link diagram in the plane, then their linking number $lk(C_1, C_2)$ is defined to be the number of positive undercrossings of $C_2$ under $C_1$ minus the number of negative undercrossings of $C_2$ under $C_1$. The linking number is well defined and symmetric (see [GS] sec. 4.5). For an $n$-strand framed pure braid $\beta$ the linking matrix $L(\beta)$ of $\beta$ is the $n \times n$ symmetric integer matrix $L$ where $L_{ij} = lk(C_i, C_j)$ for $i \neq j$ and equals the framing coefficient of $C_i$ for $i = j$. Similarly, one can define the linking matrix of any ordered oriented framed generic link diagram in the plane.

Figure 7. Pure braid for $E(n)$. The large bands represent $6n - 2$ strands and all framings equal $-2$, except for the rightmost strand with framing $-n$.

Figure 8. Crossing signs in an oriented link diagram.
Remark (2.3.1). If \( r \in \mathbb{R}^n \) corresponds to \( \beta \) a framed pure braid then \( A(r) = L(\beta) \). This follows from [W1], section 1 and [GS], p. 125. We note that orientations/conventions fixed agree with both [W1] and [GS].

Any pure braid \( \beta \in P_n \) can be written as a product of Dehn twists about simple closed curves in \( \Omega_n \). Thus, we will need these three steps:

Step I. Given a pure braid \( \beta \) resulting from a single Dehn twist, determine the corresponding Artin presentation.

Step II. Compose two Artin presentations.

Step III. Correct Framings.

Remark (2.3.2). Again, Step III is necessary since when going from a framed pure braid (where framings are not canonically included) to an Artin presentation (where framings are canonically included) an ad hoc framing correction must be made at some point.

We describe these in detail.

Step I. First, \( \pi_1(\Omega_n, p_0) \) has canonical generators. Figure 9 shows \( \Omega_{22} \) with basepoint \( p_0 \) and the generator \( x_{21} \) (the other generators are similar; see also [W1] p. 225 and p. 244). Also depicted in Figure 9 are basepoints on the boundary components \( \partial_1, \ldots, \partial_{22} \) (as referred to in section 2.1).

We use two examples to illustrate this step. For the first example, take the Dehn twist depicted in Figure 10 about the oriented simple closed curve \( D_1 \) (for

Figure 9. \( \Omega_{22} \) with basepoints \( p_0, \ldots, p_{22} \) on boundary components \( \partial_0, \ldots, \partial_{22} \). Also indicated is a generator \( x_{21} \) of \( \pi_1(\Omega_{22}, p_0) \).

Figure 10. \( \Omega_{22} \) with an oriented simple closed curve \( D_1 \) and a small segment laid across it.
Figure 11. $\Omega_{22}$ with a loop representing $r_{11}, \ldots, r_{20}$.

the moment ignore the small segment laid across $D_1$). Usually one would take a cylinder neighborhood $S^1 \times [-1, 1]$ of $D_1$ in $\Omega_{22}$ and replace it with a twisted version (often a cut along $D_1$ takes place) according to some fixed orientation convention (see, for example, [GS] p.295). Following the motivation set forth in section 2.1, we prefer to realize the Dehn twist canonically as an isotopy of $\Omega_{22}$ in $\mathbb{R}^3$ as follows. Start with a copy of $\Omega_{22}$ (as in Figure 10) laying flat on the (possibly imaginary) table in front of you and a small cylinder neighborhood $N = S^1 \times [-1, 1]$ of $D_1$ in $\Omega_{22}$. The inner boundary curve of $N$ bounds a compact disk with 10 holes denoted $\Omega'_{10}$. Slowly raise $\Omega_{22}$ up off the table and while doing so grab $\Omega'_{10}$ and slowly rotate it clockwise about a central point (with the cylinder neighborhood $N$ stretching like rubber) one complete revolution. If one pictures the paths traced out by the center points of the 22 punctures in $\Omega_{22}$ during this Dehn twist, one immediately sees the pure braid obtained from Figure 7 with $n = 2$ by just taking the ‘$-1$’ box on strands 11–20 and taking the remaining strands to be trivial. This Dehn twist, realized as an isotopy, gives a self diffeomorphism $h$ of $\Omega_{22}$ that is fixed on $\partial \Omega_{22}$, namely the time 1 map of the isotopy. As discussed above in section 2.1 and [W1] pp. 243–244, the automorphism $h_*$ of $\pi_1(\Omega_{22}, p_0) \cong F_{22}$ induced by $h$ is of the form $x_i \mapsto r_i^{-1}x_i r_i$ for some words $r_i$ and $r = \langle x_1, \ldots, x_{22} \mid r_1, \ldots, r_{22} \rangle$ is our desired Artin presentation. The word $r_i$ is nontrivial ($\neq 1$) only for $i = 11, \ldots, 20$ and these are all equal to one another. To compute $r_{11}$, say, lay a straight segment across $D_1$ in front of $\partial_{11}$ as in Figure 10 and follow the segment through the isotopy above. After the isotopy, add two oriented edges to the isotoped segment: one from $p_0$ to the upper endpoint and one from the lower endpoint to $p_0$ as in Figure 11; the word in $\pi_1(\Omega_{22}, p_0)$ represented by this oriented loop is $r_{11} = x_{20}^{-1}x_{19}^{-1}\cdots x_{11}^{-1}$.

We note two important points concerning the above example. First, it conveyed the orientation convention of Dehn twists used here, namely grab the inner compact disk with holes and twist it in the direction of the arrow on the curve one is twisting about. Second, the small segment laid across $D_1$ formed the ‘meat’ of the relations and only crossed $D_1$ once. When computing $r_i$, in general, one must choose this segment to traverse all occurrences of the curve
one is twisting about between a nice path (usually a straight line segment or a small isotopy of one) from $p_0$ to $p_i$. This is shown in the following example.

For this example, take the Dehn twist depicted in Figure 12. The automorphism of $F_{22}$ is clearly the identity on $x_1, \ldots, x_{10}, x_{22}$. Figure 13 shows the loop representing both words $r_{11} = r_{21} = x_{21}^{-1} x_{11}^{-1}$ (as the reader can verify using the two small segments in Figure 12 that cross $D_{24}$ once). The more interesting relations are $r_{12}, \ldots, r_{20}$, which are all equal to one another. To compute these one must use a segment that crosses $D_{24}$ twice, such as the middle segment in Figure 12. The resulting loop is shown in Figure 14 and represents the word $x_{21} x_{11} x_{21}^{-1} x_{11}^{-1}$. This completes Step I.

**Step II.** Our data is two Artin presentations $r, r'$ arising from Dehn twists about $D, D'$ with corresponding $h, h', h'_h, h'_g$. Then, the composite Artin presentation $r'' = r' \circ r$ is obtained using the formula (see [W1], p.245):

$$r''_i = r'_i \cdot h'_g(r_i).$$

Step II is impractical by hand when the presentations are not small and use of a computer algebra system, such as MAGMA, is invaluable.

**Step III.** Our data now is a framed pure braid $\beta$ and an Artin presentation $r'$ resulting from repeated applications of Steps I and II. One also has the

---

**Figure 12.** $\Omega_{22}$ with an oriented simple closed curve $D_{24}$ and three small segments laid across it.

**Figure 13.** $\Omega_{22}$ with a loop representing $r_{11}$ and $r_{21}$. 
matrices $L(\beta)$ and $A(r')$ which differ only possibly on their diagonals. One corrects (see Remark (2.3.2) and Section 2.1) using the simple rule:

let $\delta_i = L(\beta)_{ii} - A(r')_{ii}$, and

let $r_i = x_i^{\delta_i} \cdot r'_i$.

The result is the Artin presentation $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ and $A(r) = L(\beta)$. We point out that when correcting framings one must multiply on the left by the corresponding $x_i^{\delta_i}$, otherwise the resulting presentation is usually not Artin. This completes Step III.

(2.4) **Artin presentation of $K3$.** Begin with the framed pure braid in Figure 7 with $n = 2$. Call this braid $\beta$ and recall that all framings equal $-2$. We need a series of Dehn twists producing $\beta$ (ignoring framings for the moment). To take care of $\beta$ (reading up from the bottom) up until the point where the two large bands first cross each other, perform Dehn twists about $D_1$, $D_2$, $D_3$, and $D_4$ (in that order!) as in Figure 10 and Figures 15–17. (It may seem that the ‘$-1$’ on the left band has been left off, but the reader should check that this is not the case.) Now we attack the brunt of $\beta$ consisting of the ‘Milnor fiber’ where the two large bands cross each other and then intertwine. For this part we will need
Figures 18 and 19 repeated in an alternating fashion. Figure 18 represents $D_5, D_7, D_9, \ldots, D_{23}$ where $D_{5+2j}, j = 0, 1, 2, \ldots, 9$, corresponds to Figure 18 with $k = j + 1$ and $k' = j + 11$. Figure 19 represents $D_6, D_8, D_{10}, \ldots, D_{22}$ where $D_{6+2j}, j = 0, 1, 2, \ldots, 8$, corresponds to Figure 19 with $k = j + 1$. Then, one performs Dehn twists about the following ordered and oriented curves: $D_5, D_6, \ldots, D_{22}, D_{23}$. The reader should check that this series of Dehn twists performs as claimed. To finish up, one twists about $D_{24}$ as in Figure 12 and then about $D_{25}$ as in Figure 20. This series of Dehn twists gives $\beta$ up to framings.

Now, using Step I from section 2.3, one writes down the Artin presentation corresponding to each of the Dehn twists in this series. We organize this data into a $25 \times 22$ array $R$ of words in $F_{22}$ where $R[i, \cdot]$ corresponds to $D_i$ (i.e. $R[i, j]$ is the $j^{th}$ relation of the $i^{th}$ Artin presentation). Assume that $R$ is initialized as the $25 \times 22$ array of identity elements in $F_{22}$. The nontrivial elements in $R$ are as follows.

\[
R[1, i] = \begin{cases} 
R[i, 1] & i = 11, \ldots, 20 \\
0 & \text{otherwise}
\end{cases}
\]

\[
R[1, i] = \begin{cases} 
x_1^{-1} x_9 x_8 x_7^{-1} x_6 x_5 x_4^{-1} x_3 x_2^{-1} x_1 x_0 \\
0 & \text{otherwise}
\end{cases}
\]
Now, the relations $R[5 - 23, i]$ lend themselves well to looping/shorthand (which we utilize especially when using MAGMA). Let $w = x_{10}^{-1}x_8^{-1} \cdots x_{11}^{-1}$ and let $w_j$ denote the first $j$ letters of $w$ read from the right for $j = 0, \ldots, 9$. For example, $w_0 = 1$ (i.e. the identity in $F_n$) and $w_2 = x_{12}^{-1}x_{11}^{-1}$. Then, $R[5, i]$, $R[7, i]$, $R[23, i]$ are defined by the following where $j = 0, 1, \ldots, 9$:
Figure 19. $\Omega_{22}$ with an oriented simple closed curve $D_\ast$.

Figure 20. $\Omega_{22}$ with an oriented simple closed curve $D_{25}$.

Also, \( R(6, i), R(8, i), R(22, i) \) are defined by the following where \( j = 0, 1, \ldots, 8 \):

\[
\begin{array}{|c|c|}
\hline
R(6 + 2j, i) & \\
\hline
i = (j + 1), \ldots, 10 & x_{j+1}x_{j+2} \cdots x_{11+j}w_j \\
\hline
i = 11+j & w_jx_{j+1}x_{j+2} \cdots x_{11+j} \\
\hline
\end{array}
\]

And the last two Artin presentations:

\[
\begin{array}{|c|c|}
\hline
R(24, i) & \\
\hline
i = 11, 21 & x_{21}^{-1}x_{11}^{-1} \\
\hline
i = 12, \ldots, 20 & x_{21}x_{11}x_{21}^{-1}x_{11}^{-1} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
R(25, i) & \\
\hline
i = 21, 22 & x_{21}x_{22} \\
\hline
\end{array}
\]
The list of Artin presentations corresponding to the series of Dehn twists
given above is complete. Now, one simply composes these 25 presentations
(with MAGMA) using a loop statement and the formula from Step II in section
2.3. Call the result of this iterated composition \( r' \). To correct the framings, one
computes the exponent sum matrix of \( r' \) (again using MAGMA) and checks the
diagonal of this matrix which is (starting from the upper left):

\[
\begin{pmatrix}
-1, \ldots, -1, 0, -1, 0, \ldots, 0, 1 \\
9 \text{ times} & 10 \text{ times}
\end{pmatrix}
\]

To make these entries all equal \(-2\), one corrects \( r' \) using Step III calling the
result \( r \). This is the desired Artin presentation for the Kummer surface \( K3 \).

After obtaining \( r \) with MAGMA, one immediately checks that the presen-
tation is in fact Artin. To do so, simply prompt MAGMA to compute the right
hand side of the Artin condition (AC). The result should be (and for our \( r \)
is) the left hand side of (AC). This is an important test, but it is also a test
that MAGMA can always carry out as the word problem in \( F_n \) is solved and
MAGMA must only freely reduce.

By construction, \( M^3(r) \) is \( S^3 \) and \( W^4(r) \) is \( K3 \). Despite the length of the
presentation \( r \) (which is given below) MAGMA readily verifies that \( \pi(r) = 1 \).
To look at \( W^4(r) \) one proceeds to \( A(r) \) which appears in Figure 21. This matrix
is even, unimodular, has 19 negative eigenvalues and 3 positive ones, hence is
\( \mathbb{Z} \)-congruent to \( 2E_8 \oplus 3H \) as expected. One is now ready to reap the rewards
of this work. The Artin presentation \( r \) can be easily and orderly investigated
with MAGMA where nothing has to be done by hand and one doesn’t need to
worry about surgery diagrams, etc. Examples appear in the following section.

The inverse matrix of \( A(r) \), which appears in Figure 22, provides the peripheral structure of the knots \( k_i \), \( i = 0, \ldots, 22 \), described at the beginning of this
section. Notice that the diagonal consists entirely of \(-2, 0, \text{ and } 2 \), which as a consequence immediately again gives Artin presentations for the appropriate
(1, \( \pm 1 \)) Dehn spheres. Further, notice that the total sum of \( A(r)^{-1} \), denoted \( s \),
equals \(-6\), another computational advantage.

The knots \( k_i \) are nontrivial only for \( i = 0, 10, 11, 21, 22 \); \( k_{10} \) and \( k_{11} \) are 52s,
\( k_{22} \) is a trefoil, and \( k_{21} \), with Alexander polynomial \( \Delta = t^4 - t^2 + 1 \), is a cable of the trefoil. However, \( k_0 \) has Alexander polynomial \( \Delta = t^8 - 2t^7 - 5t^5 + 13t^4 - \ldots \)
and is off the usual knot tables; its 2, 3, 4, 5 torsion is given by (29), (13, 13),
(15, 435), (251, 251).

It seems curious that here the only non-fibered knots are \( k_{10} \) and \( k_{11} \), precisely where the pair of 3s appears off the diagonal in \( A(r)^{-1} \) (Figure 22); see also the end of section 2.1.

As \( R_{22} \) is a group, one may wish to compute \( r^{-1} \). To do so, one performs the
same series of Dehn twists as for \( r \) but in the reverse order and with reverse orientation. One must repeat Step I for all of these Dehn twists and the work
is equivalent to the work involved with getting \( r \). After doing so, one compares
the lengths of the relations in \( r \) and \( r^{-1} \) which appear below. (We use \#r
to denote the total length of all relations.) We note that shorter presentations
are not necessarily more useful computationally, especially with MAGMA, as
one quickly finds.
−2 −1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1
−1 −2 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1
−1 −1 −2 −1 −1 −1 −1 −1 −1 −1 1 1 1
−1 −1 −1 −2 −1 −1 −1 −1 −1 −1 1 1 1
−1 −1 −1 −1 −1 −2 −1 −1 −1 −1 1 1 1
−1 −1 −1 −1 −1 −1 −1 −1 −1 −1 1 1 1
−1 1 1 1 1 1 1 1 1 1 1 1 1 2
−1 1 1 1 1 1 1 1 1 1 1 1 1 2
−1 1 1 1 1 1 1 1 1 1 1 1 1 2
−1 1 1 1 1 1 1 1 1 1 1 1 1 2
−1 1 1 1 1 1 1 1 1 1 1 1 1 2
−1 1 1 1 1 1 1 1 1 1 1 1 1 2

Figure 21. $A(r)$ for $r$ representing the Kummer surface.

<table>
<thead>
<tr>
<th>Relation</th>
<th>$r$</th>
<th>$r^{-1}$</th>
<th>Relation</th>
<th>$r$</th>
<th>$r^{-1}$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>176</td>
<td>12</td>
<td>252</td>
<td>502</td>
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<tr>
<td>2</td>
<td>131</td>
<td>403</td>
<td>13</td>
<td>247</td>
<td>501</td>
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<td>15</td>
<td>231</td>
<td>499</td>
</tr>
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<td>16</td>
<td>220</td>
<td>498</td>
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<td>1291</td>
<td>17</td>
<td>207</td>
<td>497</td>
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<td>18</td>
<td>192</td>
<td>496</td>
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<td>137</td>
<td>1723</td>
<td>19</td>
<td>175</td>
<td>495</td>
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<tr>
<td>9</td>
<td>138</td>
<td>1936</td>
<td>20</td>
<td>156</td>
<td>494</td>
</tr>
<tr>
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<td>644</td>
<td>2126</td>
<td>21</td>
<td>529</td>
<td>573</td>
</tr>
<tr>
<td>11</td>
<td>258</td>
<td>108</td>
<td>22</td>
<td>5</td>
<td>383</td>
</tr>
</tbody>
</table>

Total Relator Length

<table>
<thead>
<tr>
<th>#r</th>
<th>#r$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4562</td>
<td>17260</td>
</tr>
</tbody>
</table>

In the following, we denote the just constructed $r, r^{-1}$ by $k3, k3^{-1}$. Let $t_1$ be the Torelli of [W1] p. 228. If we multiply $k3^{-1}$ “at 20 by $t_1$” [W1] p.227,
Figure 22. The inverse matrix $A(r)^{-1}$ providing the peripheral structures of the knots $k_0, \ldots, k_{22}$.

i.e. if we take the Artin presentation $r \in R_{22}$ where $r_i$ equals 1 for $i < 20$ and equals $t_1$ written in the variables $x_{20}, x_{21}, x_{22}$ for $i = 20, 21, 22$ and multiply it by $k3^{-1}$, we obtain an Artin presentation, which we denote by $k3^{-1}t_{120}$, then $\pi$ remains trivial and all knot groups stay the same except $G_0$ whose Alexander polynomial changes from $\Delta = t^8 - 2t^7 - 5t^5 + 13t^4 - \ldots \to \Delta = t^{10} - 8t^9 + 14t^8 - 2t^7 - 13t^6 + 15t^5 - \ldots$ (both polynomials are irreducible and the new 2, 3, 4, 5 torsions are given by (9), (65, 65), (3, 3, 9), (899, 899)).

Assuming the latter homotopy 3-sphere is actually $S^3$, we have two a priori different smooth structures on the same underlying topological 4-manifold. (Recall that the Torelli preserve $A(r)$ and Freedman’s theorem holds if the boundaries are the same).

Do these smooth structures differ due to, say, the arguments of Fintushel-Stern [FS]?

To obtain another Artin presentation for the $K3$ surface, which we denote by $k3$ and with inverse $k3^{-1}$, we take the pure braid in Figure 7 with $n = 2$ and modify it by an isotopy (the same modification applies to $E(n)$ in general). Take the portion of $C_{21}$ that crosses under the right large band and intertwines with the left large band and simply slide it down to the bottom of the braid and then, using the (not drawn) trivial segments that close the braid, slide it around to the top of the braid. The result is shown in Figure 23. Of course, the framings for this braid are the same as before. Following Steps I-III above we obtain $k3$. The isotopy of the braid preserved the order of the strands and hence the
matrix $A(r)$ for this new presentation is exactly the same as before (Figure 21). For these Artin presentations we have $\#k_3 = 6994$ and $\#k_3^{-1} = 4398$. We note that $k_3^{-1}$ is the shortest of the four Artin presentations given here for the Kummer surface.

3. Examples

Thanks to the computer friendly, simple presentations of knot groups and their peripheral structures in AP theory, examples therein need not be laboriously constructed: they just need to be systematically discovered with MAGMA. Due to the ‘conical’, universal structure of AP theory, at least in principle this can at least be done in a systematic, orderly, complete way. Thus, from the beginning AP theory, due to the fact, e.g. that framings need not be put in by hand, automatically and easily yields many of the known interesting examples of classical 3-manifold and knot theory: old and new. From the simplest definition of Poincaré’s homology 3–sphere to examples pertaining to the Cabling conjecture [GAS]. Specifically, at the very beginning [W1] AP theory easily yields cosmetic surgery examples, Luft-Sjerve spheres with fixed point free involutions, failure of Property R in general for $\mathbb{Z}$-homology 3–spheres, in particular giving boundaries of Mazur manifolds, and nontrivial knots in homotopy 3–spheres with trivial Alexander polynomial, a phenomenon first discovered by Seifert in the early 1930s.
Using the just constructed Artin presentations of the $K3$ surface, we continue illustrating this natural, canonical flow of instructive examples.

If $G$ is a group, by $ab(G, n)$ we denote the abelianizations of the subgroups of index $\leq n$ (up to conjugation) and we use MAGMA notation, e.g., $ab(G, 4) = 1[0], 2[7, 0], 4[2, 2, 0], 4[0, 0], 0$, means that $G$ abelianizes to $Z$ and has, up to conjugation, one subgroup of index $2$ which abelianizes to $Z_2 \times Z$, no subgroups of index $3$, and two subgroups of index $4$ abelianizing to $Z_2 \times Z_2 \times Z$ and $Z \times Z$, respectively.

By, say, $k3^{-1}st24$, we denote the Artin presentation in $\mathcal{R}_{24}$ obtained by not changing $r_i$ of $k3^{-1}$ for $i \leq 22$ and setting $r_{23} = x_{23}$ and $r_{24} = x_{24}$. It is clear (see end of previous section) what, say, $k3^{-1}st24t_3.22 \in \mathcal{R}_{24}$ should be. By $x_i^n r_i$ we denote the Artin presentation where $x_i$ is changed to $x_i^n r_i$. The Torelli $t_1, t_2, t_3 \in \mathcal{R}_3$ and $T'_i \in \mathcal{R}_4$ are as in [W1] pp.228,229,231. Furthermore, $\Delta_i$ denotes the Alexander polynomial of $k_i$.

I. Regarding the Cabling Conjecture [GAS] in general. Consider $\Sigma^3(r)$ where $r = x_{22}^{-1}k3^{-1}st24t_3.22 \in \mathcal{R}_{24}$ ($#r = 17301$); $\pi(r)$ has a balanced (non-Artin) presentation with just three generators:

\[ \langle a, b, c \mid c^2 = bcb, (cbc)^{-1}ab^6(cbc)^{-1}a^{-1}b^{-1}cbc = b^{-6}ab^6 = b^{-2}(cbc)^{-1}a^{-1}cbbc^{-6}a(ba)^2cbc \}, \]

and is therefore $\pi$-prime in the sense of [GAS], however, the $(1, -1)$ Dehn sphere of the knot $k_{21}$ has fundamental group isomorphic to $I(120) \ast \pi_1(\Sigma(2, 3, 11))$. Question: is this Dehn sphere homeomorphic to $\Sigma(2, 3, 5) \# \Sigma(2, 3, 11)$?

The knot $k_{23}$ has the same Alexander polynomial as that of the granny knot in $S^3$, but their knot groups differ since they have different $ab(r, 5)s$.

The $(1, 1)$ Dehn sphere of the knot $k_3$, where $\Delta_3 = t^2 - t + 1$, is simply connected and so $\Sigma^3(r)$ is a $(1, \pm 1)$ Dehn sphere of a knot $k$ in a homotopy $3$–sphere with Alexander polynomial $\Delta = t^2 - t + 1$, but whose group $G$ has a different $ab(r, 3)$ than that of the trefoil and is presented by:

\[ G = \langle a, b, c \mid bcb = cb^2c, b(a, (b^{-1}a) \hat{\ast} \left( b^2(cbc)^{-1}c(cbc)^{-1} \right)) \rangle. \]

Here, recall that in MAGMA notation $(x, y) = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. The homology sphere $\Sigma^3(r)$ is the quotient space of a free regular action of $I(120)$ on an $M^3$ with $H_1(M^3, Z) = \mathbb{Z}_{12}^3$ and $ab(r, 15) = ab(I(120), 15)$, however their $ab(r, 20)s$ differ. The Casson invariant, $\lambda(\Sigma^3(r))$, of $\Sigma^3(r)$ is $\pm 1$.

Question: is $G$ a knot group of $S^3$?

II. Tinkering with our Artin presentations for $K3$ seems to give an abundance of $\mathbb{Z}$–homology $3$–spheres with nontrivial knots where Property $R$ fails, i.e. $G / \langle l \rangle = \mathbb{Z}$ where $l$ is the longitude.

i) $k_{10}, k_{11}$ of $\Sigma^3(r)$ where $r = x_{18}^{-1}x_{20}^{-1}k3^{-1}t_2.1 \in \mathcal{R}_{22}$ ($#r = 17916$).

ii) $k_{00}, k_{22}$ of $\Sigma^3(r)$ where $r = x_{14}^{-1}k3^{-1}T_3.19 \in \mathcal{R}_{22}$ ($#r = 37009$).

iii) $k_{15}, k_{22}$ of $\Sigma^3(r)$ where $r = x_{20}^{-1}k3^{-1}t_3.20 \in \mathcal{R}_{22}$ ($#r = 44913$).

iv) $k_{10}, k_{11}$ of $\Sigma^3(r)$ where $r = x_{18}^{-1}k3^{-1}t_1.9 \in \mathcal{R}_{22}$ ($#r = 48643$). Here, $ab(G_{10}, 5) = 1[0], \ldots, 5[0], 5[0, 0], 5[0, 0, 0], 5[2, 0, 0], 5[28371, 0]$. The fundamental group of its $(1, 1)$ Dehn sphere has one single subgroup of index 5 and
it abelianizes to \( \mathbb{Z}_{28371} \). Such large finite numbers have not appeared before in computations in AP theory. What does their appearance mean?

v) The simplest example seems to be \( k_{22} \) of \( \Sigma^3(r) \) where \( r = x_{20}^{-1}k3^{-1}st23t3.21 \in \mathcal{R}_{23} \) (\( #r = 27628 \)). Here \( \pi(r) \) and \( G_{22} \) are presented by:

\[
\pi(r) = \langle a, b \mid (aba)^3 = (bab)^2, (ba)^3 = (a^{-1}bab)^2 \rangle,
\]

\[
G_{22} = \langle a, b \mid (aba)^3 = (ba)^2 (bab)^{-1} (ab)^2 \rangle.
\]

As is well known, the falsity of Property R, i.e. \( G/\langle l \rangle = \mathbb{Z} \), implies that the Alexander polynomial is trivial; we also obtain an abundance of nontrivial knots with trivial Alexander polynomials in homotopy 3-spheres (such examples were first discovered by Seifert in the early 1930's): let

\[
r = x_{20}^{-1}k3^{-1}st24t3.22 \in \mathcal{R}_{24} \) (\( #r = 17301 \)),
\]

then \( \Sigma^3(r) \) is simply connected and \( \Delta_20 = 1 \) but \( ab (G_{20}, 5) = 1[0], \ldots, 5[0] \), and \( 5[3, 15, 0] \) repeated 5 times; let \( r = x_{2}k3t2.20 \in \mathcal{R}_{22} \) (\( #r = 11101 \)), then \( \Sigma^3(r) \) is simply connected and \( \Delta_1 = \Delta_2 = 1 \) but \( ab (G_1, 5) = ab (G_{12}, 5) = 1[0], \ldots, 5[0], 5[0, 0, 0], 5[3, 3, 0] \). Here \( G_{12} \) is presented by:

\[
\langle a, b, c \mid (a^{-1}, c), (c, b) \rangle (a, b) c = b = (c^{-1}, a^{-1}) (b, c^{-1}) \rangle (a, b) (c^{-1}, a^{-1}) \rangle.
\]

Question: is \( G_{12} \) a knot group of \( S^3 \)?

III. If \( r = k3^{-1}t3.20 \in \mathcal{R}_{22} \) (\( #r = 44550 \)), then \( \Delta_1 \equiv 1 \) and \( \Delta_2 \equiv 1 \) but \( G_1 \) and \( G_2 \) are not isomorphic since their \( ab(, 5) \)s differ. However, both of their \( (1, 1) \) Dehn spheres are simply connected. This illustrates in a different way the phenomenon that 'far away' knots in homotopy 3-spheres can have homeomorphic \( (1, 1) \) Dehn spheres [Br].

Unlike with the Donaldson matrices \( E_8, \varphi_{4n}, \) etc., with \( K3 \) we obtain a much larger amount of knots with \( \Delta \equiv 1 \). Is this related to the 'softness' of \( K3 \) as a Calabi-Yau manifold?

### 4. The manifolds \( W^4(r) \)

We have answered in the affirmative whether all elliptic surfaces \( E(n) \) appear as \( W^4(r) \)s. An open problem is whether every smooth, compact, connected, simply-connected 4-manifold \( X^4 \) with a connected, simply-connected boundary \( \partial X^4 = M^3 \) is a \( W^4(r) \). (See [GS] p.344 for a related problem).

In dimension 3, AP theory obtains all closed, orientable, connected 3-manifolds and there seem to be no great conceptual difficulties on the horizon in obtaining all Seiberg-Witten invariants of 3-manifolds in AP theory [L], [T] pp.viii,115. Unlike in the simplicial combinatorial case, in AP theory the same purely group-theoretic data that determines the 3-manifold, namely \( r \), also canonically and holographically determines the 4-manifold. Hence, developing 3-dimensional Seiberg-Witten theory in this, its correct, ultimate arena, holds greater promise in further developing also the outstanding open 4-dimensional theory in AP theory.

Similar arguments hold for studying the smoothings of a 4-manifold, à la Fintushel-Stern [FS], using the action of the Torelli, thus generalizing their important work. We remark that, if the 3D Poincaré conjecture were true, then
by Freedman’s theorem the relation between the Torelli action and smoothings would become even more direct, purely group-theoretic and pristine, perhaps too much so.

Relevant to all of the above is that although finitely presented group theory is considered a difficult subject, the undeniable metamathematical similarities of AP theory with braid theory, holographic dessins ˇdenfant theory, as well as numerous genuine analogies with Modern Physics, give hope for a definitive, realistic, computer approachable, holographic, and universal approach to $X^4$ theory [D] p.69, [W2], [W3].

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Department of Mathematics
University of Maryland
College Park, MD 20742
USA
jsc3@math.umd.edu
hew@math.umd.edu

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