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Research Statement

My primary research lies in low-dimensional topology. I also study geometric, algebraic, and real algebraic aspects of topology.

1 Current Research

1.1 The Topological Fundamental Group

The concept of a topology for the fundamental group appears to have originated with Hurewicz (1935), but it has been actively studied only for the last ten years. The classical fundamental group of a pointed topological space $X$ naturally inherits the quotient topology from the space of based loops in $X$ endowed with the compact-open topology. The resulting space, called the topological fundamental group, can contain more information than the fundamental group, as shown by Fabel [F1]. An immediate question is to understand when the topological fundamental group is discrete. John McCarthy and I proved (2009) a folklore theorem stating that if $X$ is a locally path connected topological space, then the topological fundamental group of $X$ is discrete if and only if $X$ is semilocally simply connected [CM]. Our work was recently cited in the preprints [B] and [F2] both of which show that multiplication is not continuous in general in the topological fundamental group.

1.2 Quotient Maps and the Fundamental Group

Arno,ld (1993) [A] asked whether an exotic $\mathbb{R}^4$ (i.e., a smooth manifold homeomorphic but not diffeomorphic to $\mathbb{R}^4$) can appear as the orbit space of a polynomial vector field on $\mathbb{R}^5$. Every exotic $\mathbb{R}^4$ appears in this way for a smooth (even analytic) vector field. In a sense, his question is asking whether exotic smoothness appears naturally in the simplest possible setting, just as chaotic behavior appears naturally in $\mathbb{R}^3$ with the Lorenz attractor. More generally, Arno,ld’s problem prompts one to ask which smooth manifolds appear as orbit spaces of smooth vector fields on a given smooth manifold. As a first step to addressing this question, Bob Gompf, John McCarthy and I showed that if the orbit space is a smooth manifold (much less is actually required), then the quotient map induces a surjection of fundamental groups [CGM]. This restricts which spaces may appear as orbit spaces of vector fields on a given smooth manifold.

1.3 The Pullback Functor on Coverings

Quillen (1978) [Q] gave a triad of algebraic equivalents for certain desirable properties of the pullback functor on coverings for a map $f: X \rightarrow \{\text{point}\}$ where $X$ is a poset (one may take $X$ to be a simplicial complex). For example, $X$ is simply connected if and only if $f^*$ is an equivalence of categories (i.e., is fully faithful and essentially surjective). One is led to seek such equivalences for a map $f: X \rightarrow Y$ where $Y$ need not be a point. Recently, McCarthy and I showed that $f^*$ is essentially injective if and only if $f_{#}$ is surjective. This result holds for reasonably nice spaces, but in more generality than simplicial complexes. Upon streamlining our proof, we uncovered an interesting question concerning the detection of $f^*$ to not be essentially injective using only finite component covers of $Y$. We are currently working to answer this question.
2 Past Research

2.1 Low-dimensional Topology: Artin Presentations

2.1.1 Background

An Artin presentation is a certain type of group presentation (defined below) that determines a unique closed orientable 3-manifold by Winkelnkemper’s open book construction with planar page [W]. González-Acuña [G-A] showed that every closed orientable 3-manifold appears in this way, thus obtaining his fundamental result: *Artin presentations characterize the fundamental groups of closed orientable 3-manifolds.*

An Artin presentation also determines a unique smooth, compact, simply connected 4-manifold, bounded by the 3D open book, by a relative open book construction. Thus, in Artin presentation theory (AP theory), a smooth 4-manifold is determined by an Artin presentation of the fundamental group of its boundary. A basic tenet of AP theory is that topological invariants of 3- and 4-manifolds should be computed group theoretically, solely in function of the discrete Artin presentation.

Historically, algebraic interpretations of topological notions have had spectacular consequences. Riemann defined the genus of an algebraic curve topologically, and this important concept was algebraized by Clebsch, Max Noether and Brill. The topological Lefschetz fixed point theorem led to Weil’s famous conjectures in number theory (later proved by Deligne). The Lefschetz hyperplane theorem became Hard Lefschetz theory by work of Weil, Grothendieck and Deligne.

Open books are real, smooth analogues of Lefschetz’s hyperplane decompositions. By González-Acuña’s work, open book theory becomes completely algebraic in dimension 3, but with finitely presented group theory instead of algebraic geometry or number theory. Therefore, it becomes imperative to study 4D smooth applications of AP theory in a purely algebraic and group theoretic manner.

By definition, an Artin presentation \( r = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle \) is a balanced presentation satisfying the equation:

\[
(r_1^{-1}x_1r_1)(r_2^{-1}x_2r_2) \cdots (r_n^{-1}x_nr_n) = x_1x_2 \cdots x_n
\]

in the free group \( F_n = \langle x_1, x_2, \ldots, x_n \rangle \). These presentations arise naturally from Artin’s theory of braids. Equation (*) is deceptively simple, yet it ties together automorphisms of \( F_n \), pure braids, closed orientable 3-manifolds, knots and links therein, and smooth, compact, simply connected 4-manifolds. Let \( M^3(r) \) and \( W^4(r) \) denote the 3- and 4-manifolds determined by \( r \). Closed 4-manifolds appear in AP theory as \( W^4(r) \) with boundary \( M^3(r) = S^3 \) (close up with a 4-handle). Every closed orientable 3-manifold is homeomorphic to some \( M^3(r) \). It is open whether every smooth, closed, simply connected 4-manifold appears in AP theory; while this is appears to be a difficult question, no counterexamples are known.

2.1.2 Nontrivial Donaldson and Seiberg-Witten Invariants

After Winkelnkemper first discovered that an Artin presentation determines a unique smooth compact 4-manifold, it was conceivable that these 4-manifolds all had trivial gauge theoretic invariants. Building on work of Harer, Kas, and Kirby, Winkelnkemper and I (2004) obtained all complex elliptic surfaces \( E(n) \) in AP theory. In particular the Kummer surface \( K3 = E(2) \) appears in AP theory. Thus, AP theory contains a nontrivial, purely group theoretic theory of Donaldson and Seiberg-Witten invariants [CW].
2.1.3 Torelli Juggling of Smooth Structures

The set of Artin presentations itself forms a group, where the group operation is defined algebraically. Contained therein is the Torelli subgroup (a subgroup of the classical Torelli subgroup for each \( n \)). Composition with Torelli provides geometric transitions that are very complicated. Using work of Akbulut, I showed in my 2004 PhD thesis that multiplication by a Torelli, a completely group theoretic operation, can change the smooth structure of a 4-manifold \([C4]\). This is an entirely new way of changing the smooth structure on an underlying topological 4-manifold.

2.1.4 Other Results in AP Theory

González-Acuña gave a group theoretic formula for the Rohlin invariant of any integral homology 3-sphere in AP theory \([G-A, CW]\). I generalized this by giving a group theoretic formula for the Casson invariant of any rational homology 3-sphere in AP theory \([C2]\). This shows that all 3D Seiberg-Witten invariants may be computed group theoretically in AP theory by a result of Lim \([L]\).

The exponent sum matrix of an Artin presentation is always symmetric. There were several known proofs of this fact, but each used topological arguments in an essential way. I proved this fact in combinatorial group theory proceeding directly from the definition of an Artin presentation \([C3]\).

2.2 Real Algebraic Topology

Similar to his question above, Arnold also asked (in personal communication with Alberto Verjovsky \([MV]\)) whether an exotic \( \mathbb{R}^4 \) can appear as a nonsingular, real algebraic subset of \( \mathbb{R}^5 \). Motivated by this question, Henry King and I (2004) investigated a more general problem, namely the noncompact analogue of a classical theorem of Seifert (1936): which smooth, proper, codimension-1 submanifolds of \( \mathbb{R}^{n+1} \) are isotopic to nonsingular, real algebraic subsets? In Seifert’s compact case, the answer was “all”, whereas the noncompact case is richer and more delicate. We showed (Corollary 1 of \([CK]\)) that this real algebraic problem is equivalent to a topological one concerning the existence of completions of a manifold pair. Our main theorem (Theorem 1 of \([CK]\)) gave a necessary and sufficient topological condition for a smooth, compact manifold \( X \) with boundary to have a codimension-1, real algebraic interior: for such an \( X \), there is a smooth, proper embedding \( X \hookrightarrow D^{n+1} \) into the standard \((n + 1)\)-disk if and only if the interior of \( X \) is diffeomorphic to a nonsingular, real algebraic subset of \( \mathbb{R}^{n+1} \); furthermore, if such an embedding exists, then \( \text{Int}(X) \) is isotopic to a nonsingular, real algebraic subset of \( \text{Int}(D^{n+1}) \approx \mathbb{R}^{n+1} \). The proof built upon previous work of Akbulut and King, which was a joy to learn from Henry while I was a graduate student, and utilized a new trick which carefully crushed some circles to points.

2.3 Geometric Topology

After Henry King and I reduced the above noncompact Seifert problem to a topological question concerning completions, we turned our attention to these completions. This led us to Siebenmann’s thesis where we discovered that one of his applications that we wanted, namely a classification of smooth, proper submanifolds of \( \mathbb{R}^{n+1} \) that are isomorphic to a disjoint union of finitely many copies of \( \mathbb{R}^n \), was both misstated and unproved (\([Si]\), Theorem 10.10, p. 117). This led to an interesting collaboration with Siebenmann \([CKS]\). In this work, we give the first general treatment of the operation connected sum at infinity (also known as end sum). This yields a very natural proof of the famous theorem of Cantrell and Stallings asserting unknottedness of any proper codimension-1 hyperplane in \( \mathbb{R}^n \) in all three manifold categories (satisfying a natural ray
unknotted unknotting hypothesis in dimension 3). This provides an elementary proof of Mazur and Brown's important Schoenflies theorem. We also obtained an elementary proof of the result Henry and I originally desired, namely a classification of multiple hyperplane embeddings. The correct classification is very naturally phrased in terms of so-called gaskets which are central to our connected sum at infinity operation. Our joint paper [CKS] also contains a proof of a folklore theorem concerning proper homotopy classes of rays and the Mittag-Leffler condition and is currently submitted for publication. Henry and I then obtained further results such as a smoothing theorem for isolated nonsmooth points of codimension-1 embeddings which is similar in spirit to both Chernavsky and Kirby's result on isolated non-flat points and Bers' result on isolated singularities of minimal surfaces. Henry and I will submit this work for publication once [CKS] is accepted.

2.4 Arctangent Identities for Pi

My first mathematical love was the existence of certain arctangent identities for pi such as Machin's infamous formula \( \pi/4 = \arctan(1/2) + \arctan(1/3) \). In my youth, I sought an efficient single-angle, rational arctangent identity for pi. As an undergraduate, I proved no such identities exist [C1] (years later I discovered the result was known by different methods since the 1890s, but my proof was short and elementary and, more importantly, I had my first experience of struggling with a problem for years with eventual success). I returned to this topic (2009) and gave a very natural proof of the same result using Gaussian integers and unique factorization [C5]. John Stillwell used my approach in his recent book [S] (see p. 168). I also gave several applications to geometry and presented a thorough history of the topic. In a slightly different direction, I recently classified so-called grade school triangles (i.e., right triangles with rational angles and rational or quadratic irrational side lengths). The answer is somewhat surprising and will appear in the Monthly [C6]. Both of the papers [C5] and [C6] contain further research problems appropriate for undergraduates. I have worked with one undergraduate at Michigan State University on such problems.

3 Future Research

3.1 SU(2) Representations of 3-manifold Groups

I plan to study a collection of related problems in this area. First, one would like find an analogue of Floer theory in AP theory. The conceptual simplicity of AP theory makes it a natural candidate for the study of the universality of Floer theory (i.e., do there exist nontrivial irreducible integral homology 3-spheres with trivial Floer homology groups?) I would also like to study mysterious factors of two appearing in the Casson invariant by replacing the classical, general Heegaard decompositions used in the definition of the Casson invariant with those coming from AP theory (an Artin presentation determines a special Heegaard decomposition of the associated 3-manifold). I am interested in applying real algebraic geometry to certain algebraic curves arising from traces of an SU(2) representation of a 3-manifold group to study the important problem: does the fundamental group of every nontrivial integral homology 3-sphere admit a nontrivial representation into SU(2)? One may ask this question for nontrivial closed 3-manifolds as well.

3.2 Minimal Genus and Exotic Rational Surfaces

I raised the following question as a postdoc at the University of Texas: can the minimal genus drop in an exotic rational surface? Known techniques for producing exotic 4-manifolds (i.e. knot surgery, log transforms, and so forth) all seem to increase the minimal genus. However, for rational surfaces there are gaps in the topological and smooth genus bounds thus suggesting that
a drop is possible. Note that this question has relevance to the study of basic classes and how they may change in an exotic manifold. AP theory contains natural surfaces, dual in a sense to canonical knots in AP theory arising from the binding in the open book construction, that I will use to study this genus question.

3.3 Orbit Spaces of Smooth Vector Fields

I will study further manifolds and spaces in general that may arise as orbit spaces of smooth vector fields on smooth manifolds. First, we ask: is such an orbit space necessarily semilocally simply connected in general? Bob Gompf, John McCarthy and I have answered this question in the affirmative in several special cases, but the general question remains open. Second, must a smooth manifold arising as the orbit space of a polynomial vector field on $\mathbb{R}^n$ be collared at infinity? Interestingly, such orbit spaces may be compact, for example all complex projective spaces arise in this way since the Hopf vector field (multiplication by $i$) on $\mathbb{C}^n$ stereographically projects to a real quadratic vector field on $\mathbb{R}^{2n-1}$. Bill Goldman has suggested possible further examples using Grassmannians. Bob Gompf has suggested the possibility of a handle-body theory for manifolds arising as orbit spaces of smooth vector fields. We would like to see if the problem can be made completely topological in terms of certain restricted handle decompositions.

References


