Example of an $\epsilon - \delta$ Proof

Section 2.3, Exercise 38

We want to prove that $\lim_{x \to 3} 3x - 7 = 2$. Just like in class, we will first have a “work” section to decide what $\delta$ needs to be and then we will write our proof.

Work:

Start with $|f(x) - L| < \epsilon$. Remember that we are trying to work toward something that looks like $|x - x_0| < \delta$, or equivalently, $-\delta < x - x_0 < \delta$. In this problem, $x_0 = 3$. Remember that there are a couple different ways to get $\delta$. You can use whichever way you like, so long as there’s no illegal math involved. On trickier problems, one method may work better than another. Until you get a feel for which way you should go, you should start with the method you are most comfortable with. If you get stuck, try a different method. I will show you 2 different (although equivalent in this case) ways to find $\delta$ for this problem.

Method 1

\[
|f(x) - L| < \epsilon \\
|(3x - 7) - 2| < \epsilon \\
|3x - 9| < \epsilon \\
|3(x - 3)| < \epsilon \\
3|(x - 3)| < \epsilon \\
|(x - 3)| < \frac{\epsilon}{3}
\]

This looks like $|x - 3| < \delta$. So, we decide to let $\delta = \frac{\epsilon}{3}$.

Method 2

\[
|f(x) - L| < \epsilon \\
|(3x - 7) - 2| < \epsilon \\
-\epsilon < (3x - 7) - 2 < \epsilon \\
-\epsilon < 3x - 9 < \epsilon \\
9 - \epsilon < 3x < 9 + \epsilon \\
\frac{9 - \epsilon}{3} < x < \frac{9 + \epsilon}{3} \\
3 - \frac{\epsilon}{3} < x < 3 + \frac{\epsilon}{3} \\
-\frac{\epsilon}{3} < x - 3 < \frac{\epsilon}{3}
\]

This looks like $-\delta < x - x_0 < \delta$. So, we see that we should pick $\delta = \frac{\epsilon}{3}$.

Now that we have our $\delta$, we can begin our proof.
Proof:

Let $\epsilon > 0$. Let $\delta = \frac{\epsilon}{3}$. For all $x$ such that $0 < |x - 3| < \delta$, the following is true.

\[
|f(x) - L| = |(3x - 7) - 2| \\
= |3x - 9| \\
= |3(x - 3)| \\
= 3|x - 3| \\
< 3\delta \\
= 3\left(\frac{\epsilon}{3}\right) \\
= \epsilon
\]

By the definition of a limit, since $|f(x) - L| < \epsilon$ whenever $0 < |x - 3| < \delta$, we see that the limit of $f(x) = 3x - 7$ as $x$ goes to 3 is 2. Thus, $\lim_{x \to 3} 3x - 7 = 2$. 

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