Cubing the Pyramid:
or
Why We Need Calculus

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Believe not everything, but only what is proven: the former is foolish, the latter the act of a sensible man.
The volume of a pyramid is

\[
\frac{1}{3} \times \text{area of base} \times \text{height}.
\]
The area of a $1 \times 1$ square
The area of a $1 \times 1$ square is $1$ square unit, by definition.
The area of a $1 \times 2$ rectangle
Rectangles

The area of a $1 \times 2$ rectangle

is $1 + 1 = 2$. 
Rectangles

The area of a $1 \times \frac{2}{3}$ rectangle
Rectangles

The area of a $1 \times \frac{2}{3}$ rectangle is $\frac{1}{3} \cdot 2 = \frac{2}{3}$.

The area of a $1 \times A$ rectangle is $A$. 
More Rectangles

Claim: Any rectangle can be cut and the pieces rearranged so that it is a $1 \times A$ rectangle, for some $A$. 
More Rectangles

First: Cut and rearrange so that the height is between 1 and 2.
Too tall:
More Rectangles

**First:** Cut and rearrange so that the height is between 1 and 2.

**Too tall:**

[Diagram of two rectangles, one blue and one red, demonstrating the height issue.]
More Rectangles

First: Cut and rearrange so that the height is between 1 and 2.
Too tall:
More Rectangles

Too short:
More Rectangles

Too short:
More Rectangles

Too short:
More Rectangles

Rectangle with height between 1 and 2.
More Rectangles

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\[ \text{Diagram:} \quad \text{Triangle} \rightarrow \text{1 x A rectangle} \]
Claim: Any triangle can be cut and rearranged into a $1 \times A$ rectangle, for some $A$. 

\begin{figure}
\centering
\includegraphics[width=0.7\textwidth]{triangle_rearrangement.png}
\caption{Diagram showing how a triangle can be rearranged into a rectangle.}
\end{figure}
Claim: Any polygon can be cut and rearranged into a $1 \times A$ rectangle, for some $A$. 
Polygons

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Polygons

Area is invariant under cutting and rearranging.

And it is the only invariant for polygons.

Finding areas of polygons is fundamentally discrete.
Cubing the pyramid?

**Hilbert:** Can a regular tetrahedron be cut and rearranged to be a cube?
Cubing the pyramid?

**Hilbert:** Can a regular tetrahedron be cut and rearranged to be a cube?

**Dehn:** No.

How to prove?
The Dehn Invariant

We need another invariant.

For each edge of a polyhedron, we measure its length.

We also measure the angle the two adjoining faces make with each other.
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**Weirdness 1:** We add angles mod 180 degrees (e.g., $225 = 45$).
Weirdness 2: For a given edge with length $\ell$ and angle $\theta$, we look at

$$\ell \otimes \theta.$$

Properties

- $a \otimes b_1 + a \otimes b_2 = a \otimes (b_1 + b_2)$
- $a_1 \otimes b + a_2 \otimes b = (a_1 + a_2) \otimes b$
The Dehn Invariant

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**Properties**

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- $a_1 \otimes b + a_2 \otimes b = (a_1 + a_2) \otimes b$

$$a \otimes 0 + a \otimes 0 = a \otimes (0 + 0) = a \otimes 0$$

so

$$a \otimes 0 = 0.$$
The Dehn Invariant: Sum $\ell \otimes \theta$ over all edges of the polyhedron.

Dehn Invariant of $\ell \times \ell \times \ell$ cube

\[
\underbrace{\ell \otimes 90 + \cdots + \ell \otimes 90}_{12} = \ell \otimes 12 \cdot 90
\]

\[
= \ell \otimes 0
\]

\[
= 0
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Dehn Invariant of tetrahedron with edge length $s$

\[
s \otimes 70.529 + \cdots + s \otimes 70.529 = s \otimes 6 \cdot 70.529
\]

\[
= s \otimes 63.173
\]
The Dehn Invariant

Claim: The Dehn Invariant is invariant under cutting (and rearranging).
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The Dehn Invariant

**Claim:** The Dehn Invariant is invariant under cutting (and rearranging).

\[ \ell_1 \otimes \theta + \ell_2 \otimes \theta = (\ell_1 + \ell_2) \otimes \theta = \ell \otimes \theta. \]

\[ s \otimes \psi + s \otimes (180 - \psi) = s \otimes 180 = 0. \]
Therefore, we cannot chop up and rearrange the tetrahedron into an easier shape in order to find its volume.

We need calculus.
Democritus: Two pyramids with congruent bases and the same heights have the same volume.
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Three pyramids of equal volume can be joined to form a triangular prism.
Beyond

**Slyder:** Volume and Dehn Invariant are the *only* invariants in 3d.

**Open:** What about higher dimensions?
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Open: What about higher dimensions?

Laczkovich, 1990 If we’re allowed crazy cuts, we can cut a circle of area 1 into 9 pieces, rearrange the pieces, and get a square of area 1.
Beyond

**Slyder:** Volume and Dehn Invariant are the *only* invariants in 3d.

**Open:** What about higher dimensions?

**Laczkovich, 1990** If we’re allowed crazy cuts, we can cut a circle of area 1 into 9 pieces, rearrange the pieces, and get a square of area 1.

**Banach-Tarski** If we’re allowed crazy cuts, we can cut a sphere of volume 1 into a finite number of pieces, rearrange the pieces, and get a sphere of volume $1,000,000,000,000,000$. 