Assignment #5

PHYS-410
Fall 2013
Mr. Scofield

Announcements

Reading

If you have not already done so please read the notes I have posted online regarding density of states and the MB velocity distribution.

Read Chapter 5, Sections 1 through 3 (to page 174). The various thermodynamic definitions and relationships can make your head spin! Important topics include: Enthalpy, Helmholtz free energy, Gibbs free energy, chemical potential, Understand what is meant by negative temperature and population inversion.

Homework Problems (from Schroeder, unless otherwise specified)

Your solutions to the problems below are due at the beginning of class, Friday, October 18

5.01 **Problem 6.22.** In most paramagnetic materials, the individual magnetic particles have more than two independent states (orientations). The number of independent states depends on the particle's angular momentum "quantum number" j, which must be a multiple of 1/2. For $j = 1/2$ there are just two independent states, as discussed in the text above and in Section 3.3. More generally, the allowed values of the z component of a particle's magnetic moment are

$$\mu_z = -j\delta_\mu,\ (-j+1)\delta_\mu,\ \ldots,\ (j-1)\delta_\mu,\ j\delta_\mu,$$

where δ_μ is a constant, equal to the difference in μ_z between one state and the next. (When the particle's angular momentum comes entirely from electron spins, δ_μ equals twice the Bohr magneton. When orbital angular momentum also contributes, δ_μ is somewhat different but comparable in magnitude. For an atomic nucleus, δ_μ is roughly a thousand times smaller.) Thus the number of states is $2j+1$. In the presence of a magnetic field B pointing in the z direction, the particle's magnetic energy (neglecting interactions between dipoles) is $-\mu_z B$.

(a) Prove the following identity for the sum of a finite geometric series:

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}.$$

(Hint: Either prove this formula by induction on n, or write the series as a difference between two infinite series and use the result of Problem 6.20(a).)

(b) Show that the partition function of a single magnetic particle is

$$Z = \frac{\sinh[b(j+\frac{1}{2})]}{\sinh\frac{b}{2}},$$

where $b = \beta\delta_\mu B$.

(c) Show that the total magnetization of a system of N such particles is

$$M = N\delta_\mu\left[(j+\tfrac{1}{2})\coth[b(j+\tfrac{1}{2})] - \tfrac{1}{2}\coth\tfrac{b}{2}\right],$$

where $\coth x$ is the hyperbolic cotangent, equal to $\cosh x/\sinh x$. Plot the quantity $M/N\delta_\mu$ vs. b, for a few different values of j.

(d) Show that the magnetization has the expected behavior as $T \to 0$.

(e) Show that the magnetization is proportional to $1/T$ (Curie's law) in the limit $T \to \infty$. (Hint: First show that $\coth x \approx \frac{1}{x} + \frac{x}{3}$ when $x \ll 1$.)

(f) Show that for $j = 1/2$, the result of part (c) reduces to the formula derived in the text for a two-state paramagnet.

5.02 Show that $\sum_{n=0}^{\infty} n x^n = \frac{x}{(x-1)^2}$.

Hint: Recall earlier you used induction to show that $f(x) \equiv \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Here you may wish to look at the derivative, df/dx and see how this relates to the sum of interest.

5.03 At absolute zero temperature the N atoms of a certain solid are arranged in a perfect (simple) cubic lattice. This represents the minimum energy configuration for these atoms. It is possible for an atom to move from a cubic lattice site to an interstitial site at the center of a unit cell (i.e., cube). This increases the energy of the solid by an amount ε. Note that the vacancy (empty lattice site) and interstitial atom need not be tied to the same unit cell – each are free to "wander" around the crystal. These vacancy/interstitial pairs are called "Frenkel" defects.

Show that in thermal equilibrium at T > 0 is the structure of this solid is not a perfect crystal, but instead contains n vacancies (and necessarily n interstitial defects) given by

$$n = Ne^{-\varepsilon/2k_BT}$$

Hint: Find the number of ways that n vacancies and n interstitials can be arranged in a lattice containing N unit cells. Find that value of n that minimizes the free energy of the solid.

5.04 **Problem 5.23.** By subtracting μN from U, H, F, or G, one can obtain four new thermodynamic potentials. Of the four, the most useful is the **grand free energy** (or **grand potential**),

$$\Phi \equiv U - TS - \mu N.$$

(a) Derive the thermodynamic identity for Φ, and the related formulas for the partial derivatives of Φ with respect to T, V, and μ.

(b) Prove that, for a system in thermal and diffusive equilibrium (with a reservoir that can supply both energy and particles), Φ tends to decrease.

(c) Prove that $\Phi = -PV$.

(d) As a simple application, let the system be a single proton, which can be "occupied" either by a single electron (making a hydrogen atom, with energy -13.6 eV) or by none (with energy zero). Neglect the excited states of the atom and the two spin states of the electron, so that both the occupied and unoccupied states of the proton have zero entropy. Suppose that this proton is in the atmosphere of the sun, a reservoir with a temperature of 5800 K and an electron concentration of about 2×10^{19} per cubic meter. Calculate Φ for both the occupied and unoccupied states, to determine which is more stable under these conditions. To compute the chemical potential of the electrons, treat them as an ideal gas. At about what temperature would the occupied and unoccupied states be equally stable, for this value of the electron concentration? (As in Problem 5.20, the prediction for such a small system is only a probabilistic one.)

Though not stated, it is clear that the volume V is assumed to be fixed.