Quantal recurrence in the infinite square well

a. Classical period:

\[ E = \frac{1}{2}mv^2 \quad \text{so} \quad v = \sqrt{2E/m} \]

and

\[ \text{distance} = \text{speed} \times \text{time}, \]

so

\[ \text{period} = \frac{\text{distance}}{\text{speed}} = \frac{2L}{\sqrt{2E/m}} = L\sqrt{2m/E}. \]

b. Quantal recurrence:

How does the initial wavefunction \( \psi(x; 0) \) change with time? Expanded the initial wavefunction into energy eigenfunctions \( \eta_n(x) \):

\[ \psi(x; 0) = \sum_{n=1}^{\infty} c_n \eta_n(x). \]

This wavefunction evolves in time to

\[ \psi(x; t) = \sum_{n=1}^{\infty} c_n e^{-iE_n t/\hbar} \eta_n(x), \quad (*) \]

where the eigenvalues are

\[ E_n = \frac{\pi^2 \hbar^2}{2ML^2} n^2 = E_1 n^2 \quad \text{for} \quad n = 1, 2, 3, \ldots. \]

The evolved wavefunction will equal the initial wavefunction whenever all of the phase factors \( e^{-iE_n t/\hbar} \) are equal to one. That is, the revival occurs at a time \( T_{\text{rev}} \) where

\[ \frac{E_n}{\hbar} T_{\text{rev}} = 2\pi \quad \text{(some integer)} \]

for all values of \( n \). Using the eigenenergy result this becomes

\[ \frac{E_1}{\hbar} T_{\text{rev}} n^2 = 2\pi \quad \text{(some integer)} \]

so the revival time is

\[ T_{\text{rev}} = \frac{2\pi \hbar}{E_1} = \frac{\hbar}{E_1} = \frac{4mL^2}{\pi \hbar}. \]

(Note that we solved this part knowing only the energy eigenvalues.)

c. What happens after one-half of this time has passed?

Evaluated at \( t = T_{\text{rev}}/2 \), formula (*) gives us

\[ \psi(x; T_{\text{rev}}/2) = \sum_{n=1}^{\infty} c_n e^{-iE_n T_{\text{rev}}/2\hbar} \eta_n(x). \]

But \( T_{\text{rev}} = h/E_1 \), so

\[ \frac{E_n T_{\text{rev}}}{2\hbar} = \frac{E_1 T_{\text{rev}}}{2\hbar} n^2 = \pi n^2 \]

1
and
\[ \psi(x; T_{\text{rev}}/2) = \sum_{n=1}^{\infty} c_n e^{-i\pi n^2} \eta_n(x). \]

Now
\[ e^{-i\pi n^2} = (-1)^n = (-1)^{\frac{n}{2}} \]
so
\[ \psi(x; T_{\text{rev}}/2) = \sum_{n=1}^{\infty} c_n (-1)^n \eta_n(x). \]

But the energy eigenfunction \( \eta_n(x) \) is even for \( n \) odd and odd for \( n \) even, so
\[ (-1)^n \eta_n(x) = -\eta_n(-x) \]
whence
\[ \psi(x; T_{\text{rev}}/2) = -\psi(-x; 0). \]

That is: After half a revival time, the initial wavefunction is flipped from left to right and turned up-side down (that is, multiplied by the physically-irrelevant overall phase factor of \(-1\)). (Note that we solved this part knowing only the energy eigenvalues and the parity of the energy eigenfunctions.)