Scaling in the stadium problem

Famed mathematician George Pólya coined the term "the inventor's paradox" in his book *How to Solve It* — "The more ambitious plan may have more chances of success." I like to phrase this principle as "It can be easier to prove a more general statement than a particular case of that statement." This solution shows the inventor's paradox at work, because rather than use an enlargement factor of 3 we use an enlargement factor of α. And rather than solve for this particular stadium potential we solve for any two dimensional potential $V(x,y)$.

The original problem is: Solve the time development problem

$$\frac{\partial\psi(x,y,t)}{\partial t} = -\frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+V(x,y)\psi(x,y,t)\right] \tag{1}$$

subject to the initial condition

$$\psi(x,y,0)=\psi_0(x,y). \tag{2}$$

We call this solution $\psi(x,y,t)$.

The new problem is: Solve the problem in potential $U(x,y)$ proportional to $V(x/\alpha,y/\alpha)$, that is

$$U(x,y)=\beta V(x/\alpha,y/\alpha), \tag{3}$$

subject to initial condition $\phi_0(x,y)$ proportional to $\psi_0(x/\alpha,y/\alpha)$, that is

$$\phi_0(x,y)=\gamma\psi_0(x/\alpha,y/\alpha). \tag{4}$$

We call this solution $\phi(x,y,t)$. Our objective is to find $\phi(x,y,t)$ in terms of $\psi(x,y,t)$.

We suspect that the answer will be simply time scaled by factor δ, that is,

$$\phi(x,y,t)=\gamma\psi(x/\alpha,y/\alpha,t/\delta). \tag{5}$$

We will have solved the problem when we've found δ in terms of α, and confirmed the conjecture above. ⟦It would also be nice, but not necessary, to find β and γ in terms of α.⟧

Now, we want the solution to

$$\frac{\partial\phi(x,y,t)}{\partial t} = -\frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+U(x,y)\phi(x,y,t)\right] \tag{6}$$

subject to the initial condition

$$\phi(x,y,0)=\phi_0(x,y). \tag{7}$$

According to our conjecture (5),

$$\frac{\partial\phi(x,y,t)}{\partial x}=\gamma\frac{\partial\psi(x/\alpha,y/\alpha,t)}{\partial x}=\gamma\frac{\partial\psi}{\partial(x/\alpha)}\frac{\partial(x/\alpha)}{\partial x}=\frac{\gamma}{\alpha}\frac{\partial\psi}{\partial(x/\alpha)} \tag{8}$$

and

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\gamma}{\alpha}\frac{\partial\psi}{\partial(x/\alpha)}\right)=\frac{\partial}{\partial(x/\alpha)}\left(\frac{\gamma}{\alpha}\frac{\partial\psi}{\partial(x/\alpha)}\right)\frac{\partial(x/\alpha)}{\partial x}=\frac{\gamma}{\alpha^2}\frac{\partial^2\psi}{(x/\alpha)^2} \tag{9}$$

so

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\gamma}{\alpha^2} \left(\frac{\partial^2 \psi}{\partial (x/\alpha)^2} + \frac{\partial^2 \psi}{\partial (y/\alpha)^2} \right). \tag{10}$$

Meanwhile, if our conjecture is correct, then

$$\frac{\partial \phi}{\partial t} = \frac{\gamma}{\delta} \frac{\partial \psi}{\partial (t/\delta)}. \tag{11}$$

Now, if ϕ satisfies (6), and our conjecture is correct, then

$$\frac{\gamma}{\delta} \frac{\partial \psi}{\partial (t/\delta)} = -\frac{i}{\hbar} \left[-\frac{\hbar^2}{2m} \frac{\gamma}{\alpha^2} \left(\frac{\partial^2 \psi}{\partial (x/\alpha)^2} + \frac{\partial^2 \psi}{\partial (y/\alpha)^2} \right) + \beta V(x/\alpha, y/\alpha) \gamma \psi(x/\alpha, y/\alpha, t/\delta) \right]. \tag{12}$$

A glance at equation (1) shows that this does indeed hold, provided that

$$\frac{\gamma}{\delta} = \frac{\gamma}{\alpha^2} = \gamma\beta. \tag{13}$$

Our conjecture has been validated.

We conclude that $\delta = \alpha^2$ — quantum time development in a stadium three times larger requires nine times as much time.

Does it makes sense that time development requires more time in a larger stadium? Absolutely: You should expect it to take longer to cross a 300 foot field than a 100 foot field. But why does it take nine times longer rather than three times longer? Keep reading!

⟦Not required, but can we find expressions for β and γ in terms of α? The expression for β is easy and follows directly from (13): it is $\beta = 1/\alpha^2$. We can find γ through the requirement of normalization:

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\phi_0(x,y)|^2 \, dx \, dy \\ &= \gamma^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\psi_0(x/\alpha, y/\alpha)|^2 \, dx \, dy \\ &= \gamma^2 \alpha^2 \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\psi_0(x/\alpha, y/\alpha)|^2 \, d(x/\alpha) \, d(y/\alpha) \right] \\ &= \gamma^2 \alpha^2 [1] \end{aligned}$$

whence $\gamma = 1/\alpha$. ⟧

What about energies? Energy eigenstate n in the original problem evolves in time like $e^{-(i/\hbar)E_n t}$ — that is, it goes back to the initial state after time $T = 2\pi\hbar/E_n$ has elapsed. In the enlarged stadium it takes nine times as much time, so the corresponding E_n must be one-ninth the original.

Does it make sense that the energy eigenvalues for the larger stadium will be smaller? Yes. The energy eigenvalues for the larger stadium are more closely spaced, and in the limit of a very large (classical) stadium the energy spacing will go to zero.

Mean energies will scale as one-ninth the original, so mean velocities will scale as one-third the original ($E = \frac{1}{2}mv^2$). Compared to the original problem, the stretched wavepacket has to move three times as far with one-third the velocity. This resolves the question we left hanging about "why does it take nine times longer rather than three times longer?"

⟦A glance at the proof shows that these results apply regardless of dimension. And if you think about our solution to the infinite square well, you'll see that these results apply there!⟧

In February 2014, student Zachary Mark pointed out another way to look at the scaling of velocity.

Suppose the stadium has width W. Any quantity whatsoever can depend only on the parameters W, $\hbar$, and m — these are the only parameters in the problem.

There is only one way to build a quantity with the dimensions of velocity from these three parameters, namely through

$$\frac{\hbar}{mW}.$$

Thus if the width increases by a factor of 3, the mean velocity must decrease by a factor of 3.

Come to think of it, you can prove a lot of these results through dimensional analysis.

Grading: This problem is more free form — there are many possible approaches — so it's harder to produce a grading one-size-fits-all grading scheme. (Some people call such a scheme a "rubric".) The general principle is that you earn 4 or 5 points for starting out and setup, then 5 or 6 points for execution. If you think of some out-of-the-box solution (as Zachary Mark did in 2014), then you earn extra credit points.