The stress-energy four-tensor

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We’ve previously discussed the flow of momentum, and our discussion resulted in the Maxwell stress tensor. In relativity we have to discuss the flow of four-momentum. The result will be the stress-energy four-tensor.

As a technical appetizer, I just want to mention that the time-space four-vector

\[ r = [ct, \vec{r}] \]  

has time derivative

\[ \frac{dr}{dt} = [c, \vec{v}] . \]

This is not a four-vector, because we’re taking the derivative with respect to laboratory time \( t \) rather than proper time \( \tau \). But we’ll soon find the result useful anyway.

Now on to the main course. Remember that the negative of the Maxwell stress tensor described the flow of electromagnetic momentum. We called the momentum density \( \vec{g} \), and the velocity of a plug of fields \( \vec{v} \), and described that momentum flow through the tensor

\[
\begin{bmatrix}
\vec{g}_x \vec{v} & \leftarrow x\text{-momentum} \\
g_y \vec{v} & \leftarrow y\text{-momentum} \\
g_z \vec{v} & \leftarrow z\text{-momentum} \\
\end{bmatrix}
\]

(\( \uparrow \uparrow \uparrow \) move in direction \( x, y, z \))

(I don’t know enough \LaTeX{} to make the brackets beautiful. You’ll just have to use some imagination.)

One lesson of relativity is that we can’t think of momentum in isolation: we have to consider also the zero component of the energy-momentum four-vector, namely energy/\( c \). But we’ve already talked about the energy density \( u \), so we fill in the zero row of this four-tensor as

\[
\begin{bmatrix}
u \vec{v}/c & \leftarrow \text{energy}/c \\
g_x \vec{v} & \leftarrow x\text{-momentum} \\
g_y \vec{v} & \leftarrow y\text{-momentum} \\
g_z \vec{v} & \leftarrow z\text{-momentum} \\
\end{bmatrix}
\]

(\( \uparrow \uparrow \uparrow \) move in direction \( x, y, z \))
An even earlier lesson of relativity is that we can’t think of space in isolation: we have to consider the zero component of the time-space four-vector, namely $ct$. Using equation (2), we extend the bottom three rows of tensor into the zero column:

| $u\vec{v}/c$ | $\leftarrow$ energy/c |
| $g_x\vec{v}$ | $\leftarrow$ x-momentum |
| $g_y\vec{v}$ | $\leftarrow$ y-momentum |
| $g_z\vec{v}$ | $\leftarrow$ z-momentum |

| move in | move in |
| direction | direction |
| $ct$ | $x, y, z$ |

Finally we fill in the zero-zero component of the four-tensor

| $u\vec{S}/c$ | $\leftarrow$ energy/c |
| $S_x/c$ | $\leftarrow$ x-momentum |
| $S_y/c$ | $\leftarrow$ y-momentum |
| $S_z/c$ | $\leftarrow$ z-momentum |

| move in | move in |
| direction | direction |
| $ct$ | $x, y, z$ |

Recognizing that the flow of energy is related to the Poynting vector $\vec{S} = u\vec{v}$ and that the momentum density is $\vec{g} = \vec{S}/c^2$, this four-tensor can be written

$$
\begin{bmatrix}
  u & \vec{S}/c \\
  S_x/c & g_x\vec{v} \\
  S_y/c & g_y\vec{v} \\
  S_z/c & g_z\vec{v}
\end{bmatrix}
$$

or, remembering the Maxwell stress tensor $\vec{T}$

$$
\begin{bmatrix}
  u & \vec{S}/c \\
  S_x/c \\
  S_y/c & -\vec{T} \\
  S_z/c
\end{bmatrix}
$$

and in this form, the four-tensor is clearly symmetric. It is called the “stress-energy four-tensor”.

The stress-energy four-tensor resolves the conundrum we raised concerning the transformation of electromagnetic energy-momentum, but the concept goes beyond electromagnetism: any flow of energy-momentum
has an associated stress-energy four-tensor. For example, a particle of mass $m$ follows the trajectory $\vec{r}(t)$ in the laboratory. Using $\vec{v} = \frac{d\vec{r}}{dt}$ and $\gamma = \frac{1}{\sqrt{1 - (\vec{v} / \vec{x})^2}}$, I think you can see for yourself that the stress-energy four-tensor is

$$
\begin{bmatrix}
\gamma mc^2 & \gamma mc \vec{v} \\
\gamma mv_x c & \gamma mv_x \vec{v} \\
\gamma mv_y c & \gamma mv_y \vec{v} \\
\gamma mv_z c & \gamma mv_z \vec{v}
\end{bmatrix}
\times \delta^{(3)}(\vec{x} - \vec{r}(t)).
$$

In Newton’s theory of gravity, mass is the source of gravity. But in Einstein’s theory of gravity, general relativity, the stress-energy four-tensor is the source of gravity. The zero-zero component above shows that mass is a source of gravity, but the other components show that energy and momentum are also sources, and that the flow of energy and momentum is also a source. This explains how light can be bent by gravity: it has zero mass but it doesn’t have zero energy or zero momentum.