Eliminating field variables: Feynman-Hibbs equation (9.63)

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Derivation of equation (9.63) in *Quantum Mechanics and Path Integrals* by Richard P. Feynman and Albert R. Hibbs (McGraw-Hill, New York, 1965) or (emended edition, 2005).

Preliminary A — Forced harmonic oscillator (one dimension)

See section 8-9. If

$$L = \frac{M}{2}\dot{x}^2 - \frac{M\omega^2}{2}x^2 + f(t)x$$

then the amplitude to go from state 0 at $t = 0$ to state 0 at $t = T$ is the path integral over relevant paths

$$G_{00} = \int \exp\left\{\frac{i}{\hbar}\left[f(t)x + \frac{M}{2}\dot{x}^2 - \frac{M\omega^2}{2}x^2\right]\right\}\mathcal{D}x$$

which we evaluated in equation (8.138) as

$$G_{00} = \exp\left\{-\frac{1}{2M\omega\hbar}\int_0^T\int_0^t f(t)f(s)e^{-i\omega(t-s)}\,ds\,dt\right\}$$

But, using the fact (see discussion at equation (6.16), and (A.12))

$$\int_0^T\int_0^t g(t,s)\,ds\,dt = \int_0^T\int_t^T g(s,t)\,ds\,dt$$

gives

$$\int_0^T\int_0^t f(t)f(s)e^{-i\omega|t-s|}\,ds\,dt = \int_0^T\int_t^T f(t)f(s)e^{-i\omega|t-s|}\,ds\,dt$$

or

$$G_{00} = \exp\left\{-\frac{1}{4M\omega\hbar}\int_0^T\int_0^T f(t)f(s)e^{-i\omega|t-s|}\,ds\,dt\right\}$$

Preliminary B — Lagrangian for interaction

According to (9.32), the interaction action (for polarization 1 only) is

$$S_{\text{int}} = \sqrt{4\pi}\int j_{1,-\mathbf{k}}a_{1,\mathbf{k}}\frac{d^3\mathbf{k}}{(2\pi)^3}$$

But $j_{1,-\mathbf{k}} = j^*_{1,\mathbf{k}}$ and $a_{1,\mathbf{k}} = a^*_{1,-\mathbf{k}}$, so

$$\begin{aligned}
S_{\text{int}} &= \sqrt{4\pi}\int j^*_{1,\mathbf{k}}a_{1,\mathbf{k}}\frac{d^3\mathbf{k}}{(2\pi)^3}\\
&= \sqrt{4\pi}\int j_{1,-\mathbf{k}}a^*_{1,-\mathbf{k}}\frac{d^3\mathbf{k}}{(2\pi)^3}\\
&= \sqrt{4\pi}\int j_{1,\mathbf{k}}a^*_{1,\mathbf{k}}\frac{d^3\mathbf{k}}{(2\pi)^3}
\end{aligned}$$

Thus

$$S_{\rm int} = \sqrt{\pi} \int (j^*_{1,\mathbf{k}} a_{1,\mathbf{k}} + j_{1,\mathbf{k}} a^*_{1,\mathbf{k}}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Using the "big box of volume Vol" idea (section 4-3, or equation (A.11))

$$\begin{aligned} S_{\rm int} &= \frac{\sqrt{\pi}}{\rm Vol} \sum_{\mathbf{k}} (j^*_{1,\mathbf{k}} a_{1,\mathbf{k}} + j_{1,\mathbf{k}} a^*_{1,\mathbf{k}}) \\ &= \sqrt{\pi} \sum_{\mathbf{k}} (\bar{j}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}} + \bar{j}_{1,\mathbf{k}} \bar{a}^*_{1,\mathbf{k}}) \end{aligned}$$

Note: We don't *have* to use this new form of $S_{\rm int}$. We could use the old $j_{1,-\mathbf{k}} a_{1,\mathbf{k}}$ form. But the new form makes $S_{1,\mathbf{k}}$ correspond to a real lagrangian, whereas in the old form only $S_{1,\mathbf{k}} + S_{1,-\mathbf{k}}$ corresponds to a real lagrangian.

Main argument

According to the first line of equation (9.63) (neglecting the ground state energy term)

$$X_{1,\mathbf{k}} = \int \exp\left\{ \frac{i}{\hbar} \int \left[\sqrt{\pi}(\bar{j}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}} + \bar{j}_{1,\mathbf{k}} \bar{a}^*_{1,\mathbf{k}}) + \frac{1}{2} \dot{\bar{a}}^*_{1,\mathbf{k}} \dot{\bar{a}}_{1,\mathbf{k}} - \frac{k^2c^2}{2} \bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}} \right] dt \right\} \mathcal{D}\bar{a}_{1,\mathbf{k}}$$

This is just "code language" for an expression in terms of the real variables

$$\bar{j}_{1,\mathbf{k}} = \frac{1}{\sqrt{2}}(J_c - iJ_s) \quad \text{and} \quad \bar{a}_{1,\mathbf{k}} = \frac{1}{\sqrt{2}}(A_c - iA_s)$$

In terms of these variables, the expression in square brackets above is

$$\left[\sqrt{\pi}(J_c A_c + J_s A_s) + \frac{1}{2}\frac{1}{2}(\dot{A}_c^2 + \dot{A}_s^2) - \frac{k^2c^2}{2}\frac{1}{2}(A_c^2 + A_s^2) \right]$$

and

$$\begin{aligned} X_{1,\mathbf{k}} &= \int \exp\left\{ \frac{i}{\hbar} \int \left[\sqrt{\pi}\, J_c A_c + \frac{1}{2}\frac{1}{2}\dot{A}_c^2 - \frac{k^2c^2}{2}\frac{1}{2}A_c^2 \right] dt \right\} \mathcal{D}A_c \\ &\quad \times \int \exp\left\{ \frac{i}{\hbar} \int \left[\sqrt{\pi}\, J_s A_s + \frac{1}{2}\frac{1}{2}\dot{A}_s^2 - \frac{k^2c^2}{2}\frac{1}{2}A_s^2 \right] dt \right\} \mathcal{D}A_s \end{aligned}$$

Using preliminary A with $f(t) = \sqrt{\pi}\, J_c(t)$, $\omega = kc$, and $M = 1/2$, this evaluates to

$$\begin{aligned} X_{1,\mathbf{k}} &= \exp\left\{ -\frac{1}{4(1/2)kc\hbar} \int_0^T \int_0^T \pi J_c(t) J_c(s) e^{-ikc|t-s|}\, ds\, dt \right\} \\ &\quad \times \exp\left\{ -\frac{1}{4(1/2)kc\hbar} \int_0^T \int_0^T \pi J_s(t) J_s(s) e^{-ikc|t-s|}\, ds\, dt \right\} \\ &= \exp\left\{ -\frac{\pi}{2kc\hbar} \int_0^T \int_0^T [J_c(t)J_c(s) + J_s(t)J_s(s)] e^{-ikc|t-s|}\, ds\, dt \right\} \end{aligned}$$

Our only remaining question: whether this agrees with the Feynman-Hibbs result of

$$X_{1,\mathbf{k}} \quad = \quad \exp\left\{-\frac{\pi}{2kc\hbar}\int_0^T\int_0^T [J_c(t) - iJ_s(t)][J_c(s) + iJ_s(s)]e^{-ikc|t-s|}\,ds\,dt\right\}$$

Well,

$$[J_c(t) - iJ_s(t)][J_c(s) + iJ_s(s)] = J_c(t)J_c(s) + J_s(t)J_s(s) + i[J_c(t)J_s(s) - J_s(t)J_c(s)]$$

but, by swapping dummy variables,

$$\int_0^T\int_0^T J_c(t)J_s(s)e^{-ikc|t-s|}\,ds\,dt = \int_0^T\int_0^T J_c(s)J_s(t)e^{-ikc|s-t|}\,dt\,ds$$

and we're done.

Note: I'm not certain that I've correctly counted modes in transforming from complex normal modes to real normal modes. Compare the situation in one dimension, where when you use real normal modes you sum over only positive k.