Kernel for the forced harmonic oscillator: Feynman-Hibbs problem 3-11

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Solution to problem 3-11 in *Quantum Mechanics and Path Integrals* by Richard P. Feynman and Albert R. Hibbs (McGraw-Hill, New York, 1965).

The only hard part is finding the classical action S_{cl}.

Theorem: The force is

$$F(t) = -m\omega^2 x(t) + f(t).$$

The classical action is defined by

$$S_{cl} = \int_{t_a}^{t_b} \left(\frac{m}{2}\dot{x}^2(t) - \frac{m\omega^2}{2} x^2(t) + f(t)x \right) dt.$$

Look at the first of these three terms using integration by parts:

$$\begin{aligned} \int_{t_a}^{t_b} \dot{x}^2\, dt &= [x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x}\, dt \\ &= [x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\frac{F}{m}\, dt \\ &= [x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\left(-\omega^2 x + \frac{f(t)}{m}\right) dt \\ &= [x\dot{x}]_{t_a}^{t_b} + \omega^2 \int_{t_a}^{t_b} x^2\, dt - \int_{t_a}^{t_b} \frac{f(t)}{m} x\, dt \end{aligned}$$

So the classical action is

$$\begin{aligned} S_{cl} &= \left[\frac{m}{2}[x\dot{x}]_{t_a}^{t_b} + \frac{m\omega^2}{2}\int_{t_a}^{t_b} x^2\, dt - \frac{1}{2}\int_{t_a}^{t_b} f(t)x\, dt \right] - \frac{m\omega^2}{2}\int_{t_a}^{t_b} x^2\, dt + \int_{t_a}^{t_b} f(t)x\, dt \\ &= \frac{m}{2}\left[x(t)\dot{x}(t)\right]_{t_a}^{t_b} + \frac{1}{2}\int_{t_a}^{t_b} f(t)x(t)\, dt. \end{aligned}$$

Classical motion: The Green's function solution for the forced SHO is usually written

$$x(t) = A'\cos\omega t + B'\sin\omega t + \frac{1}{m\omega}\int_{t_a}^{t} f(s)\sin\omega(t-s)\, ds,$$

but for our purposes it is easier to write it as

$$x(t) = A\sin\omega(t-t_a) + B\sin\omega(t_b - t) + \frac{1}{m\omega}\int_{t_a}^{t} f(s)\sin\omega(t-s)\, ds.$$

The velocity is

$$\dot{x}(t) = \omega\left[A\cos\omega(t-t_a) - B\cos\omega(t_b-t) + \frac{1}{m\omega}\int_{t_a}^{t} f(s)\cos\omega(t-s)\, ds \right].$$

Initial and final values: Defining

$$\begin{aligned} X_S &= \frac{1}{m\omega}\int_{t_a}^{t_b} f(s)\sin\omega(t_b - s)\,ds \\ X_C &= \frac{1}{m\omega}\int_{t_a}^{t_b} f(s)\cos\omega(t_b - s)\,ds \end{aligned}$$

gives the initial and final values

$$\begin{aligned} x_a &= B\sin\omega T & x_b &= A\sin\omega T + X_S \\ \dot{x}(t_a) &= \omega[A - B\cos\omega T] & \dot{x}(t_b) &= \omega[A\cos\omega T - B + X_C] \end{aligned}$$

whence

$$B = \frac{x_a}{\sin\omega T} \qquad A = \frac{x_b - X_S}{\sin\omega T}.$$

First part of action: From the above,

$$\begin{aligned} [x(t)\dot{x}(t)]_{t_a}^{t_b} &= x_b\omega[A\cos\omega T - B + X_C] - x_a\omega[A - B\cos\omega T] \\ &= \omega A(x_b\cos\omega T - x_a) + \omega B(-x_b + x_a\cos\omega T) + \omega x_b X_C \\ &= \frac{\omega}{\sin\omega T}\left[x_b^2\cos\omega T - x_b x_a - x_b X_S\cos\omega T + x_a X_S - x_b x_a + x_a^2\cos\omega T + x_b X_C\sin\omega T\right] \\ &= \frac{\omega}{\sin\omega T}\left[(x_b^2 + x_a^2)\cos\omega T - 2x_b x_a + x_b(X_C\sin\omega T - X_S\cos\omega T) + x_a X_S\right]. \end{aligned}$$

But (using $\sin(A - B) = \sin A\cos B - \cos A\sin B$)

$$\begin{aligned} X_C\sin\omega T - X_S\cos\omega T &= \frac{1}{m\omega}\int_{t_a}^{t_b} f(s)\cos\omega(t_b - s)\,ds\sin\omega T - \frac{1}{m\omega}\int_{t_a}^{t_b} f(s)\sin\omega(t_b - s)\,ds\cos\omega T \\ &= \frac{1}{m\omega}\int_{t_a}^{t_b} f(s)\left[\cos\omega(t_b - s)\sin\omega T - \sin\omega(t_b - s)\cos\omega T\right]\,ds \\ &= \frac{1}{m\omega}\int_{t_a}^{t_b} f(s)\left[\sin\omega(T - t_b + s)\right]\,ds \\ &= \frac{1}{m\omega}\int_{t_a}^{t_b} f(s)\sin\omega(s - t_a)\,ds \end{aligned}$$

so

$$\begin{aligned} \frac{m}{2}\left[x(t)\dot{x}(t)\right]_{t_a}^{t_b} &= \frac{m\omega}{2\sin\omega T}\Bigg[(x_b^2 + x_a^2)\cos\omega T - 2x_b x_a \\ &\qquad + \frac{x_b}{m\omega}\int_{t_a}^{t_b} f(s)\sin\omega(s - t_a)\,ds \\ &\qquad + \frac{x_a}{m\omega}\int_{t_a}^{t_b} f(s)\sin\omega(t_b - s)\,ds\Bigg]. \end{aligned}$$

Second part of action: Meanwhile,

$$
\begin{aligned}
&\frac{1}{2}\int_{t_a}^{t_b} f(t)x(t)\,dt \\
&\quad= \frac{1}{2}\left[A\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)\,dt + B\int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\,dt + \frac{1}{m\omega}\int_{t_a}^{t_b} f(t)\int_{t_a}^{t} f(s)\sin\omega(t-s)\,ds\,dt\right] \\
&\quad= \frac{1}{2\sin\omega T}\Bigg[\,(x_b - X_S)\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)\,dt + x_a\int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\,dt \\
&\qquad\qquad + \frac{\sin\omega T}{m\omega}\int_{t_a}^{t_b} f(t)\int_{t_a}^{t} f(s)\sin\omega(t-s)\,ds\,dt\,\Bigg] \\
&\quad= \frac{1}{2\sin\omega T}\Bigg[\,x_b\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)\,dt \\
&\qquad + x_a\int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\,dt \\
&\qquad - \frac{1}{m\omega}\int_{t_a}^{t_b} f(s)\sin\omega(t_b-s)\,ds\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)\,dt \\
&\qquad + \frac{\sin\omega T}{m\omega}\int_{t_a}^{t_b} f(t)\int_{t_a}^{t} f(s)\sin\omega(t-s)\,ds\,dt\,\Bigg].
\end{aligned}
$$

Conclusion: Adding the first and second parts,

$$
\begin{aligned}
S_{cl} = \frac{m\omega}{2\sin\omega T}\Bigg[\,&(x_b^2 + x_a^2)\cos\omega T - 2x_b x_a \\
&+ \frac{2x_b}{m\omega}\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)\,dt \\
&+ \frac{2x_a}{m\omega}\int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\,dt \\
&- \frac{1}{m^2\omega^2}\int_{t_a}^{t_b} f(s)\sin\omega(t_b-s)\,ds\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)\,dt \\
&+ \frac{\sin\omega T}{m^2\omega^2}\int_{t_a}^{t_b} f(t)\int_{t_a}^{t} f(s)\sin\omega(t-s)\,ds\,dt\,\Bigg]. \qquad (1)
\end{aligned}
$$

The form of Feynman and Hibbs: The above expression for S_{cl} is, in my opinion, the cleanest and easiest to understand. Feynman and Hibbs, however, use a different form for the last two lines. Our objective is to prove that

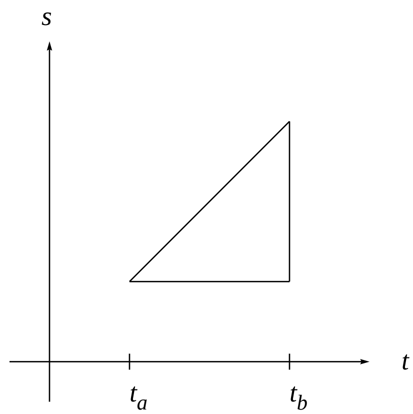
$$
\begin{aligned}
&\int_{t_a}^{t_b} f(s)\sin\omega(t_b-s)\,ds\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)\,dt - \sin\omega T\int_{t_a}^{t_b} f(t)\int_{t_a}^{t} f(s)\sin\omega(t-s)\,ds\,dt \\
&\qquad = 2\int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\int_{t_a}^{t} f(s)\sin\omega(s-t_a)\,ds\,dt. \qquad (2)
\end{aligned}
$$

Using the trigonometric difference formula $\sin(b-a)\sin(t-s) = \sin(b-s)\sin(t-a) - \sin(b-t)\sin(s-a)$ (proved in the appendix), we have

$$\sin\omega T \sin\omega(t-s) = \sin\omega(t_b - s)\sin\omega(t-t_a) - \sin\omega(t_b - t)\sin\omega(s-t_a).$$

Thus

$$\begin{aligned} & \sin\omega T \int_{t_a}^{t_b} f(t) \int_{t_a}^{t} f(s)\sin\omega(t-s)\, ds\, dt \\ = & \int_{t_a}^{t_b} f(t)\sin\omega(t-t_a) \int_{t_a}^{t} f(s)\sin\omega(t_b-s)\, ds\, dt - \int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\int_{t_a}^{t} f(s)\sin\omega(s-t_a)\, ds\, dt. \end{aligned}$$



Now, for any function $g(t,s)$, we have

$$\int_{t_a}^{t_b}\int_{t_a}^{t} g(t,s)\, ds\, dt = \int_{t_a}^{t_b}\int_{s}^{t_b} g(t,s)\, dt\, ds = \int_{t_a}^{t_b}\int_{t}^{t_b} g(s,t)\, ds\, dt$$

(First step through diagram, second through swapping the dummy variable names s and t.)

So the left-hand side of equation (2) is

$$\begin{aligned} & + \int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\int_{t_a}^{t} f(s)\sin\omega(s-t_a)\, ds\, dt \\ & + \int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\int_{t}^{t_b} f(s)\sin\omega(s-t_a)\, ds\, dt \\ & + \int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\int_{t_a}^{t} f(s)\sin\omega(s-t_a)\, ds\, dt \\ & - \int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\int_{t}^{t_b} f(s)\sin\omega(s-t_a)\, ds\, dt \\ = \; & 2\int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)\int_{t_a}^{t} f(s)\sin\omega(s-t_a)\, ds\, dt, \end{aligned}$$

which is the right-hand side.

Appendix: A trigonometric difference formula

Using first $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$, and then $\cos(A+B) = \cos A \cos B - \sin A \sin B$, we have

$$\begin{aligned}
\sin(b-a)\sin(t-s) &= \tfrac{1}{2}[\cos(b-a-t+s) - \cos(b-a+t-s)] \\
&= \tfrac{1}{2}[\cos(b-t+s-a) - \cos(b-s+t-a)] \\
&= \tfrac{1}{2}[\cos(b-t)\cos(s-a) - \sin(b-t)\sin(s-a) \\
&\qquad - \cos(b-s)\cos(t-a) + \sin(b-s)\sin(t-a)].
\end{aligned}$$

But now, applying $\cos A \cos B = \cos(A-B) - \sin A \sin B$ to the two cosine terms above, we have

$$\begin{aligned}
&\cos(b-t)\cos(s-a) - \cos(b-s)\cos(t-a) \\
= \; &\cos(b-t-s+a) - \sin(b-t)\sin(s-a) - \cos(b-s-t+a) + \sin(b-s)\sin(t-a) \\
= \; &-\sin(b-t)\sin(s-a) + \sin(b-s)\sin(t-a)
\end{aligned}$$

whence

$$\sin(b-a)\sin(t-s) = \sin(b-s)\sin(t-a) - \sin(b-t)\sin(s-a).$$