Scattering wave function: Feynman-Hibbs problem 6-13

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Solution to problem 6-13 in *Quantum Mechanics and Path Integrals* by Richard P. Feynman and Albert R. Hibbs (McGraw-Hill, New York, 1965).

Begin with equation (6-61):

$$\psi(\mathbf{R}_b, t_b) = e^{(i/\hbar)\mathbf{p}_a\cdot\mathbf{R}_b} e^{-(i/\hbar)E_a t_b} - \frac{i}{\hbar}\int_0^{t_b}\int^{\mathbf{r}_c} K_0(\mathbf{R}_b, t_b; \mathbf{r}_c, t_c) V(\mathbf{r_c}, t_c) e^{(i/\hbar)\mathbf{p}_a\cdot\mathbf{r}_c} e^{-(i/\hbar)E_a t_c}\, d^3\mathbf{r}_c\, dt_c. \quad (1)$$

Combine with the expression for the three-dimensional free-particle propagator (derived from equation 3-3),

$$K_0(\mathbf{R}_b, t_b; \mathbf{r}_c, t_c) = \left[\frac{m}{2\pi i\hbar(t_b - t_c)}\right]^{3/2} \exp\frac{im(\mathbf{R}_b - \mathbf{r}_c)^2}{2\hbar(t_b - t_c)}, \quad (2)$$

to make (for time-independent potentials)

$$\begin{aligned}\psi(\mathbf{R}_b, t_b) &= e^{(i/\hbar)\mathbf{p}_a\cdot\mathbf{R}_b} e^{-(i/\hbar)E_a t_b} \\ &\quad -\frac{i}{\hbar}\int_0^{t_b}\int^{\mathbf{r}_c}\left[\frac{m}{2\pi i\hbar(t_b - t_c)}\right]^{3/2} \exp\frac{im(\mathbf{R}_b - \mathbf{r}_c)^2}{2\hbar(t_b - t_c)} V(\mathbf{r_c}) e^{(i/\hbar)\mathbf{p}_a\cdot\mathbf{r}_c} e^{-(i/\hbar)E_a t_c}\, d^3\mathbf{r}_c\, dt_c. \quad (3)\end{aligned}$$

Collect the time dependence to find that the second line above is

$$-\frac{i}{\hbar}\int^{\mathbf{r}_c} V(\mathbf{r_c}) e^{(i/\hbar)\mathbf{p}_a\cdot\mathbf{r}_c}\left\{\int_0^{t_b}\left[\frac{m}{2\pi i\hbar(t_b - t_c)}\right]^{3/2} \exp\frac{im(\mathbf{R}_b - \mathbf{r}_c)^2}{2\hbar(t_b - t_c)} e^{-(i/\hbar)E_a t_c}\, dt_c\right\} d^3\mathbf{r}_c. \quad (4)$$

We wish to evaluate the time integral — the one within curly brackets. Use the definition $r_{bc}^2 = (\mathbf{R}_b - \mathbf{r}_c)^2$ to write this as

$$\int_0^{t_b}\left[\frac{m}{2\pi i\hbar(t_b - t_c)}\right]^{3/2} \exp\frac{imr_{bc}^2}{2\hbar(t_b - t_c)} e^{-(i/\hbar)E_a t_c}\, dt_c. \quad (5)$$

This integral is ripe for the substitution

$$x^2 = \frac{mr_{bc}^2}{2\hbar(t_b - t_c)}, \qquad t_c = t_b - \frac{mr_{bc}^2}{2\hbar x^2}, \quad (6)$$

where x is real (because $0 \le t_c \le t_b$) and dimensionless. As t_c goes from 0 to t_b,

$$x \text{ goes from } \left[\frac{mr_{bc}^2}{2\hbar t_b}\right]^{1/2} \text{ to } \infty.$$

Note that

$$\begin{aligned} 2x\, dx &= \frac{mr_{bc}^2}{2\hbar(t_b - t_c)^2}\, dt_c \\ 2\left[\frac{mr_{bc}^2}{2\hbar(t_b - t_c)}\right]^{1/2} dx &= \frac{mr_{bc}^2}{2\hbar(t_b - t_c)^2}\, dt_c \\ 2\frac{1}{r_{bc}}\frac{m}{2\hbar}\, dx &= \left[\frac{m}{2\hbar(t_b - t_c)}\right]^{3/2}\, dt_c. \end{aligned}$$

Carrying out this substitution, the integral is

$$\frac{1}{(\pi i)^{3/2}}\frac{1}{r_{bc}}\frac{m}{\hbar}e^{-(i/\hbar)E_a t_b}\int_{x_b}^{\infty} e^{ix^2}e^{(i/\hbar)E_a(mr_{bc}^2/2\hbar)/x^2}\,dx \tag{7}$$

where

$$x_b \equiv \left[\frac{mr_{bc}^2}{2\hbar t_b}\right]^{1/2}.$$

Using $E_a = p_a^2/2m$, write this expression as

$$\frac{1}{(\pi i)^{3/2}}\frac{1}{r_{bc}}\frac{m}{\hbar}e^{-(i/\hbar)E_a t_b}\int_{x_b}^{\infty} e^{ix^2}e^{i(p_a r_{bc}/2\hbar)^2/x^2}\,dx. \tag{8}$$

This integral is of the form

$$\int_{x_b}^{\infty}\exp(ia/x^2 + ibx^2)\,dx$$

with a and b real and positive. In general, the evaluation of this integral involves the error function erf(x). However in the case that $x_b = 0$ the integral has the simple value

$$\int_0^{\infty}\exp(ia/x^2 + ibx^2)\,dx = \sqrt{\frac{i\pi}{4b}}\exp(i2\sqrt{ab}).$$

Thus, in the limit that

$$\frac{mr_{bc}^2}{2\hbar t_b} \to 0, \tag{9}$$

the expression (8) becomes

$$\frac{1}{(\pi i)^{3/2}}\frac{1}{r_{bc}}\frac{m}{\hbar}e^{-(i/\hbar)E_a t_b}\left[\sqrt{\frac{i\pi}{4}}\exp(ip_a r_{bc}/\hbar)\right] = \frac{1}{2\pi i}e^{-(i/\hbar)E_a t_b}e^{(i/\hbar)p_a r_{bc}}\frac{1}{r_{bc}}\frac{m}{\hbar}. \tag{10}$$

Note that in the limit (9), it is not sufficient to say "t_b is very large". One must say "t_b is large compared to…" compared to what? Compared to something with the dimensions of time, and in particular, large compared to $mr_{bc}^2/2\hbar$.

Now, going back, we find that expression (4) is equal to

$$-\frac{m}{2\pi\hbar^2}e^{-(i/\hbar)E_a t_b}\int^{\mathbf{r}_c}\frac{1}{r_{bc}}e^{(i/\hbar)p_a r_{bc}}V(\mathbf{r_c})e^{(i/\hbar)\mathbf{p}_a\cdot\mathbf{r}_c}\,d^3\mathbf{r}_c, \tag{11}$$

subject to the proviso that limit (9) holds for *all* values of r_{bc} where $V(\mathbf{r}_c)$ is non-negligible – that is, subject to the proviso that

$$\frac{mR_b^2}{2\hbar t_b} \to 0. \tag{12}$$

Finally, substitution back into (3) produces

$$\psi(\mathbf{R}_b, t_b) = e^{-(i/\hbar)E_a t_b}\left[e^{(i/\hbar)\mathbf{p}_a\cdot\mathbf{R}_b} - \frac{m}{2\pi\hbar^2}\int^{\mathbf{r}_c}\frac{1}{r_{bc}}e^{(i/\hbar)p_a r_{bc}}V(\mathbf{r_c})e^{(i/\hbar)\mathbf{p}_a\cdot\mathbf{r}_c}\,d^3\mathbf{r}_c\right].$$