Harmonic oscillator amplitudes: Feynman-Hibbs problem 8-1

Dan Styer, Oberlin College Physics Department, Oberlin, Ohio 44074
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Solution to parts of problem 8-1 in *Quantum Mechanics and Path Integrals* by Richard P. Feynman and Albert R. Hibbs (McGraw-Hill, New York, 1965).

Use M for the mass of the particle, so that m can be a summation index.

The left-hand side of equation (8-24) is

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g^*(x_2)K(x_2,T;x_1,0)f(x_1)\,dx_1\,dx_2 = \left(\frac{M\omega}{\pi\hbar}\right)^{1/2}\left(\frac{M\omega}{2\pi i\hbar\sin\omega T}\right)^{1/2} \tag{1}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(M\omega/2\hbar)(x_2-b)^2}e^{-(M\omega/2\hbar)(x_1-a)^2}\exp\left\{\frac{iM\omega}{2\hbar\sin\omega T}[(x_2^2+x_1^2)\cos\omega T-2x_2x_1]\right\}dx_1\,dx_2.$$

First, change variables to the (dimensionless) scaled lengths

$$\bar{x}_1=\sqrt{\frac{M\omega}{2\hbar}}\,x_1,\quad \bar{a}=\sqrt{\frac{M\omega}{2\hbar}}\,a,\quad \text{etc.} \tag{2}$$

(This step is not necessary, but it makes subsequent manipulations easier.) This gives equation (1) equal to

$$\left(\frac{2}{\pi^2 i\sin\omega T}\right)^{1/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left\{-(\bar{x}_2-\bar{b})^2-(\bar{x}_1-\bar{a})^2+i\frac{\cos\omega T}{\sin\omega T}(\bar{x}_2^2+\bar{x}_1^2)-i\frac{2}{\sin\omega T}\bar{x}_2\bar{x}_1]\right\}d\bar{x}_1\,d\bar{x}_2.$$

Now drop the bar notation, and rearrange to

$$\left(\frac{2}{\pi^2 i\sin\omega T}\right)^{1/2}e^{-(a^2+b^2)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left\{\left(-1+i\frac{\cos\omega T}{\sin\omega T}\right)(x_1^2+x_2^2)+2(ax_1+bx_2)-i\frac{2}{\sin\omega T}x_1x_2\right\}dx_1\,dx_2$$

or

$$\left(\frac{2}{\pi^2 i\sin\omega T}\right)^{1/2}e^{-(a^2+b^2)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left\{\frac{1}{\sin\omega T}\left[ie^{i\omega T}(x_1^2+x_2^2)+2\sin\omega T(ax_1+bx_2)-2ix_1x_2\right]\right\}dx_1\,dx_2. \tag{3}$$

This gaussian integral could be evaluated directly, but it is easier to first execute a 45° rotation of axes to the new variables

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}(x_1'-x_2') \\ x_2 &= \frac{1}{\sqrt{2}}(x_1'+x_2'). \end{aligned}$$

In these variables, the argument of the exponential in equation (3) is

$$\begin{aligned} \{\quad\} &= \frac{1}{\sin\omega T}\left[ie^{i\omega T}({x_1'}^2+{x_2'}^2)+\sqrt{2}\sin\omega T(ax_1'-ax_2'+bx_1'+bx_2')-i({x_1'}^2-{x_2'}^2)\right] \\ &= \frac{1}{\sin\omega T}\left[i(e^{i\omega T}-1){x_1'}^2+i(e^{i\omega T}+1){x_2'}^2+\sqrt{2}\sin\omega T(a+b)x_1'+\sqrt{2}\sin\omega T(-a+b)x_2'\right] \end{aligned}$$

In these variables the two-dimensional integral in (3) becomes just a product of two one-dimensional integrals, and it is readily evaluated to

$$\left(\frac{\pi \sin \omega T}{-i(e^{i\omega T}-1)}\right)^{1/2} \exp\left\{-\frac{2(a+b)^2 \sin \omega T}{4i(e^{i\omega T}-1)}\right\} \left(\frac{\pi \sin \omega T}{-i(e^{i\omega T}+1)}\right)^{1/2} \exp\left\{-\frac{2(a-b)^2 \sin \omega T}{4i(e^{i\omega T}+1)}\right\}.$$

Straightforward manipulation shows that this equals

$$\frac{\pi \sin \omega T}{(1-e^{i2\omega T})^{1/2}} \exp\left\{ i \sin \omega T \left[\frac{(a^2+b^2)e^{i\omega T}+2ab}{e^{i2\omega T}-1}\right]\right\}$$

so, multiplying by the prefactor, equation (3) equals

$$\left(\frac{2i \sin \omega T}{e^{i2\omega T}-1}\right)^{1/2} e^{-(a^2+b^2)} \exp\left\{\frac{i \sin \omega T}{e^{i2\omega T}-1}[(a^2+b^2)e^{i\omega T}+2ab]\right\}.$$

More straightforward manipulation shows that

$$\frac{2i \sin \omega T}{e^{i2\omega T}-1} = e^{-i\omega T}$$

so equation (3) equals

$$e^{-i\omega T/2} e^{-(a^2+b^2)} \exp\left\{(e^{-i\omega T}/2)[(a^2+b^2)e^{i\omega T}+2ab]\right\}$$

or, reverting to the bar notation for scaled variables in equation (2),

$$\exp\left\{-i\frac{\omega T}{2} - \frac{\bar{a}^2+\bar{b}^2}{2} + \bar{a}\bar{b}e^{-i\omega T}\right\}$$

Expanding the bar notation our final result for equation (1) is

$$\exp\left\{-i\frac{\omega T}{2} - \frac{M\omega}{4\hbar}(a^2+b^2-2abe^{-i\omega T})\right\},$$

as claimed by Feynman and Hibbs in their equation (8-27).

Continuing, equation (8-27) is

$$e^{-i\omega T/2} \exp\left\{-\frac{M\omega}{4\hbar}(a^2+b^2-2abe^{-i\omega T})\right\} = \sum_{n=0}^{\infty} f_n^*(b) f_n(a) e^{-(i/\hbar)E_n T}$$

The left-hand side is

$$\begin{aligned}
& e^{-i\omega T/2} \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{M\omega}{4\hbar}\right)^m (a^2+b^2-2abe^{-i\omega T})^m \\
= \quad & e^{-i\omega T/2} \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{M\omega}{4\hbar}\right)^m \left(\sum_{n=0}^{m} \frac{m!}{(m-n)!n!}(a^2+b^2)^{m-n}(-2abe^{-i\omega T})^n\right) \\
= \quad & e^{-i\omega T/2} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \left(-\frac{M\omega}{4\hbar}\right)^m \frac{1}{(m-n)!n!}(a^2+b^2)^{m-n}(-2ab)^n e^{-in\omega T}
\end{aligned}$$

But for any summand $f(m,n)$, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m} f(m,n) = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} f(m,n) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} f(n+l,n)$$

so the reexpression of the left-hand side of (8.27) continues as

$$\begin{aligned}
& e^{-i\omega T/2}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\left(-\frac{M\omega}{4\hbar}\right)^{n+l}\frac{1}{l!n!}(a^2+b^2)^l(-2ab)^n e^{-in\omega T} \\
= & \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{M\omega}{4\hbar}\right)^{n}(-2ab)^n\left[\sum_{l=0}^{\infty}\left(-\frac{M\omega}{4\hbar}\right)^{l}\frac{1}{l!}(a^2+b^2)^l\right]e^{-i(n+1/2)\omega T} \\
= & \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{M\omega ab}{2\hbar}\right)^{n}\left[\sum_{l=0}^{\infty}\left(-\frac{M\omega}{4\hbar}\right)^{l}\frac{1}{l!}(a^2+b^2)^l\right]e^{-i(n+1/2)\omega T} \\
= & \exp\left\{-\frac{M\omega(a^2+b^2)}{4\hbar}\right\}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{M\omega ab}{2\hbar}\right)^{n}e^{-i(n+1/2)\omega T}
\end{aligned}$$

Comparing this to equation (8.27), we see that

$$E_n = \hbar\omega(n+1/2)$$

and that

$$f_n^*(b)f_n(a) = \exp\left\{-\frac{M\omega(a^2+b^2)}{4\hbar}\right\}\frac{1}{n!}\left(\frac{M\omega ab}{2\hbar}\right)^{n}$$

whence

$$f_n(a) = \left(\frac{M\omega}{2\hbar}\right)^{n/2}\frac{a^n}{\sqrt{n!}}\exp\left\{-\frac{M\omega a^2}{4\hbar}\right\}$$