Diffraction through a sharp-edged slit: Feynman-Hibbs section 3-3

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Notes on section 3-3 in *Quantum Mechanics and Path Integrals* by Richard P. Feynman and Albert R. Hibbs (emended edition, 2005).

The unnormalized probability density, given by Feynman-Hibbs equation (3.40), is

$$P(x') = \frac{m}{2\pi\hbar(T+t')}\left(\frac{1}{2}[C(u_+) - C(u_-)]^2 + \frac{1}{2}[S(u_+) - S(u_-)]^2\right) \tag{1}$$

where

$$u_\pm = \frac{(x' - Vt') \pm b(1+t'/T)}{\sqrt{(\pi\hbar t'/m)(1+t'/T)}}. \tag{2}$$

Following Feynman-Hibbs, define $b_1 = b(1+t'/T)$ and $\Delta x_2 = \hbar t'/mb$. Also define $\bar{x} = x' - Vt'$: this is x relative to the mean position. In terms of these variables,

$$P(\bar{x}) = \frac{t'}{T}\frac{1}{4\pi\Delta x_2 b_1}\left([C(u_+) - C(u_-)]^2 + [S(u_+) - S(u_-)]^2\right) \tag{3}$$

where

$$u_\pm = \frac{\bar{x} \pm b_1}{\sqrt{\pi\Delta x_2 b_1}}. \tag{4}$$

Plotting: Figure 3-6 in the 1965 edition purports to plot this unnormalized probability density for the three cases

	b_1	$b_1/\Delta x_2$	Δx_2
(a)	1.5	15	0.1
(b)	0.4	1	0.4
(c)	0.1	1/15	1.5

Lacking any direction from the book, I plotted the expression using $t' = T$.

This task is accomplished by the Excel spreadsheet SharpSlit.xls. This uses the Fresnel integral functions in the XNUMBERS add-in package, downloadable from

http://digilander.libero.it/foxes/Functions.htm

Caution! These functions take only positive arguments, so you must use, for example

SIGN(x)*Fresnel_cos(ABS(x))

This spreadsheet shows that all three parts of figure 3-6 in the 1965 edition have errors, although figure (a) is the worst. This is not surprising given that when the manuscript was prepared plots like these were made by hand and through looking up values in Jahnke and Emde. (Abramowitz and Stegun was published in 1964.)

Normalization: The expression for $P(\bar{x})$ above is not normalized. It doesn't even have the proper dimensions for a probability density! A normalized version can be produced easily. Define the normalization constant A through

$$P(\bar{x}) = A\left([C(u_+) - C(u_-)]^2 + [S(u_+) - S(u_-)]^2\right), \tag{5}$$

so that

$$1/A = \int_{-\infty}^{+\infty} [C(u_+) - C(u_-)]^2 + [S(u_+) - S(u_-)]^2\, d\bar{x}. \tag{6}$$

But

$$\begin{aligned}
& [C(u_+) - C(u_-)]^2 + [S(u_+) - S(u_-)]^2 \\
= & \left[\int_{u_-}^{u_+} \cos(\tfrac{1}{2}\pi t^2)\, dt\right]^2 + \left[\int_{u_-}^{u_+} \sin(\tfrac{1}{2}\pi t^2)\, dt\right]^2 \\
= & \int_{u_-}^{u_+}\int_{u_-}^{u_+} \cos(\tfrac{1}{2}\pi t^2)\cos(\tfrac{1}{2}\pi s^2)\, ds\, dt + \int_{u_-}^{u_+}\int_{u_-}^{u_+} \sin(\tfrac{1}{2}\pi t^2)\sin(\tfrac{1}{2}\pi s^2)\, ds\, dt \\
= & \int_{u_-}^{u_+}\int_{u_-}^{u_+} \cos(\tfrac{1}{2}\pi(t^2 - s^2))\, ds\, dt.
\end{aligned}$$

Define the dimensionless variables

$$\breve{x} = \frac{\bar{x}}{\sqrt{\pi\Delta x_2 b_1}} \qquad \breve{b} = \frac{b_1}{\sqrt{\pi\Delta x_2 b_1}}, \tag{7}$$

so that

$$1/A = \sqrt{\pi\Delta x_2 b_1}\int_{-\infty}^{+\infty}\int_{\breve{x}-\breve{b}}^{\breve{x}+\breve{b}}\int_{\breve{x}-\breve{b}}^{\breve{x}+\breve{b}} \cos(\tfrac{1}{2}\pi(t-s)(t+s))\, ds\, dt\, d\breve{x}. \tag{8}$$

Now translate the origin by defining $\bar{t} = t - \breve{x}$, $\bar{s} = s - \breve{x}$:

$$1/A = \sqrt{\pi\Delta x_2 b_1}\int_{-\infty}^{+\infty}\int_{-\breve{b}}^{\breve{b}}\int_{-\breve{b}}^{\breve{b}} \cos(\tfrac{1}{2}\pi(\bar{t}-\bar{s})(\bar{t}+\bar{s}+2\breve{x}))\, d\bar{s}\, d\bar{t}\, d\breve{x}. \tag{9}$$

Now the integral over $\breve{x}$ passes under the integrals over $\bar{s}$ and $\bar{t}$. We need to evaluate

$$\begin{aligned}
\int_{-\infty}^{+\infty} \cos\alpha(\beta + 2\breve{x})\, d\breve{x} &= \Re e\left\{\int_{-\infty}^{+\infty} e^{i\alpha(\beta+2\breve{x})}\, d\breve{x}\right\} \\
&= \Re e\left\{e^{i\alpha\beta}\int_{-\infty}^{+\infty} e^{i2\alpha\breve{x}}\, d\breve{x}\right\} \\
&= \Re e\left\{e^{i\alpha\beta} 2\pi\delta(2\alpha)\right\} \\
&= \cos(\alpha\beta) 2\pi\delta(2\alpha).
\end{aligned}$$

So

$$
\begin{aligned}
1/A &= \sqrt{\pi \Delta x_2 b_1} \int_{-\breve{b}}^{\breve{b}} \int_{-\breve{b}}^{\breve{b}} \cos(\tfrac{1}{2}\pi(\bar{t}-\bar{s})(\bar{t}+\bar{s})) 2\pi\delta(\pi(\bar{t}-\bar{s}))\, d\bar{s}\, d\bar{t} \\
&= \sqrt{\pi \Delta x_2 b_1} \int_{-\pi\breve{b}}^{\pi\breve{b}} \int_{-\pi\breve{b}}^{\pi\breve{b}} \cos(\tfrac{1}{2}(x-y)(x+y)/\pi) 2\pi\delta(x-y)\, \frac{dy}{\pi}\, \frac{dx}{\pi} \\
&= \sqrt{\pi \Delta x_2 b_1} \frac{2}{\pi} \int_{-\pi\breve{b}}^{\pi\breve{b}} \cos(0)\, dx \\
&= \sqrt{\pi \Delta x_2 b_1} \frac{2}{\pi} 2\pi\breve{b} = 4b_1
\end{aligned}
$$

Thus, the normalized probability density is

$$
P(\bar{x}) = \frac{1}{4b_1} \left([C(u_+) - C(u_-)]^2 + [S(u_+) - S(u_-)]^2 \right) \tag{10}
$$