The forced harmonic oscillator: Feynman-Hibbs section 8-9

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Derivation of results in section 8-9 of *Quantum Mechanics and Path Integrals* by Richard P. Feynman and Albert R. Hibbs (McGraw-Hill, New York, 1965).

Part I: Transition amplitude from 0 to 0. Evaluate G_{00} from equations (8-137), (8-17), and the form of classical action equal to (3-66) given by equation (1) of my document "Kernel for the forced harmonic oscillator", obtaining equation (8-138).

$$
\begin{aligned}
G_{00} &= e^{(i/\hbar)E_0 T} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi_0(x_b) K(x_b, T; x_a, 0) \phi_0(x_a) \, dx_a \, dx_b \\
&= e^{(i/\hbar)(\hbar\omega/2)T} \left(\frac{M\omega}{\pi\hbar} \right)^{1/2} \left(\frac{M\omega}{2\pi i\hbar \sin\omega T} \right)^{1/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(M\omega/2\hbar)(x_b^2 + x_a^2)} e^{iS_{cl}/\hbar} \, dx_a \, dx_b \\
&= e^{i\omega T/2} \frac{M\omega}{\pi\hbar} \frac{1}{\sqrt{2i \sin\omega T}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\{ \quad \} \, dx_a \, dx_b
\end{aligned}
$$

where

$$
\begin{aligned}
\{ \quad \} = &+ \left[-\frac{M\omega}{2\hbar} + \frac{iM\omega}{2\hbar} \frac{\cos\omega T}{\sin\omega T} \right] (x_b^2 + x_a^2) \\
&- \frac{iM\omega}{\hbar \sin\omega T} x_b x_a \\
&+ \frac{i}{\hbar \sin\omega T} x_b \int_0^T \gamma(t) \sin\omega t \, dt \\
&+ \frac{i}{\hbar \sin\omega T} x_a \int_0^T \gamma(t) \sin\omega(T-t) \, dt \\
&- \frac{i}{2\hbar M\omega \sin\omega T} \int_0^T \gamma(t) \sin\omega t \, dt \int_0^T \gamma(t) \sin\omega(T-t) \, dt \\
&+ \frac{i}{2\hbar M\omega} \int_0^T \int_0^t \gamma(t)\gamma(s) \sin\omega(t-s) \, ds \, dt.
\end{aligned}
$$

Define

$$
\begin{aligned}
F &= \int_0^T \gamma(t) \sin\omega t \, dt \\
G &= \int_0^T \gamma(t) \cos\omega t \, dt \\
H &= \int_0^T \gamma(t) \sin\omega(T-t) \, dt = (\sin\omega T)G - (\cos\omega T)F
\end{aligned}
$$

so that

$$
\begin{aligned}
\{\ \ \} = &+ \frac{iM\omega}{2\hbar\sin\omega T}e^{i\omega T}(x_b^2 + x_a^2) \\
&- \frac{iM\omega}{\hbar\sin\omega T}x_b x_a \\
&+ \frac{iF}{\hbar\sin\omega T}x_b \\
&+ \frac{iH}{\hbar\sin\omega T}x_a \\
&- \frac{iFH}{2\hbar M\omega\sin\omega T} + \frac{i}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt.
\end{aligned}
$$

Use the "diagonalization trick" to switch to rotated variables

$$
\begin{aligned}
x_a &= \frac{1}{\sqrt{2}}(x'_a - x'_b) \\
x_b &= \frac{1}{\sqrt{2}}(x'_a + x'_b),
\end{aligned}
$$

whence

$$
\begin{aligned}
\{\ \ \} = &+ \frac{iM\omega}{2\hbar\sin\omega T}e^{i\omega T}({x'_a}^2 + {x'_b}^2) \\
&- \frac{iM\omega}{2\hbar\sin\omega T}({x'_a}^2 - {x'_b}^2) \\
&+ \frac{iF}{\sqrt{2}\,\hbar\sin\omega T}(x'_a + x'_b) \\
&+ \frac{iH}{\sqrt{2}\,\hbar\sin\omega T}(x'_a - x'_b) \\
&- \frac{iFH}{2\hbar M\omega\sin\omega T} + \frac{i}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt \\
= &+ \frac{iM\omega}{2\hbar\sin\omega T}(e^{i\omega T} - 1){x'_a}^2 \\
&+ \frac{iM\omega}{2\hbar\sin\omega T}(e^{i\omega T} + 1){x'_b}^2 \\
&+ \frac{i(F+H)}{\sqrt{2}\,\hbar\sin\omega T}x'_a \\
&+ \frac{i(F-H)}{\sqrt{2}\,\hbar\sin\omega T}x'_b \\
&- \frac{iFH}{2\hbar M\omega\sin\omega T} + \frac{i}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
G_{00} = &\ e^{i\omega T/2}\frac{M\omega}{\pi\hbar}\frac{1}{\sqrt{2i\sin\omega T}}\exp\left\{-\frac{iFH}{2\hbar M\omega\sin\omega T} + \frac{i}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt\right\} \\
&\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\{C'x'^2_a + D'x'^2_b + A'x'_a + B'x'_b\}\,dx'_a\,dx'_b.
\end{aligned}
$$

This integral immediately factorizes into two one-dimensional gaussian integrals. Its value is

$$
\begin{aligned}
& \left(\pi\frac{-2\hbar\sin\omega T}{iM\omega(e^{i\omega T}-1)}\right)^{1/2}\exp\left\{\frac{(F+H)^2/2\hbar^2\sin^2\omega T}{4iM\omega(e^{i\omega T}-1)/2\hbar\sin\omega T}\right\}\\
& \times\left(\pi\frac{-2\hbar\sin\omega T}{iM\omega(e^{i\omega T}+1)}\right)^{1/2}\exp\left\{\frac{(F-H)^2/2\hbar^2\sin^2\omega T}{4iM\omega(e^{i\omega T}+1)/2\hbar\sin\omega T}\right\}\\
= & \frac{2\pi i\hbar\sin\omega T}{M\omega(e^{i\omega T}-1)^{1/2}(e^{i\omega T}+1)^{1/2}}\\
& \times\exp\left\{\frac{(F+H)^2}{4i\hbar M\omega\sin\omega T(e^{i\omega T}-1)}\right\}\exp\left\{\frac{(F-H)^2}{4i\hbar M\omega\sin\omega T(e^{i\omega T}+1)}\right\}\\
= & \frac{2\pi i\hbar\sin\omega T}{M\omega(e^{i2\omega T}-1)^{1/2}}\\
& \times\exp\left\{\frac{1}{4i\hbar M\omega\sin\omega T}\left[\frac{(F+H)^2}{e^{i\omega T}-1}+\frac{(F-H)^2}{e^{i\omega T}+1}\right]\right\}\\
= & \frac{2\pi i\hbar\sin\omega T}{M\omega e^{i\omega T/2}(e^{i\omega T}-e^{-i\omega T})^{1/2}}\\
& \times\exp\left\{\frac{1}{4i\hbar M\omega\sin\omega T}\left[\frac{(F+H)^2(e^{i\omega T}+1)+(F-H)^2(e^{i\omega T}-1)}{e^{i2\omega T}-1}\right]\right\}\\
= & \frac{2\pi i\hbar\sin\omega T}{M\omega e^{i\omega T/2}(2i\sin\omega T)^{1/2}}\\
& \times\exp\left\{\frac{1}{4i\hbar M\omega\sin\omega T}\left[\frac{2(F^2+H^2)e^{i\omega T}+4FH}{e^{i\omega T}2i\sin\omega T}\right]\right\}\\
= & e^{-i\omega T/2}\frac{\pi\hbar}{M\omega}\sqrt{2i\sin\omega T}\\
& \times\exp\left\{\frac{1}{4i\hbar M\omega\sin\omega T}\left[\frac{2(F^2+H^2)e^{i\omega T}+4FH}{e^{i\omega T}2i\sin\omega T}\right]\right\}\\
= & e^{-i\omega T/2}\frac{\pi\hbar}{M\omega}\sqrt{2i\sin\omega T}\exp\left\{\frac{F^2+H^2+2FHe^{-i\omega T}}{-4\hbar M\omega\sin^2\omega T}\right\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
G_{00} = & \exp\left\{-\frac{iFH}{2\hbar M\omega\sin\omega T}+\frac{i}{2\hbar M\omega}\int_0^T\int_0^t\gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt\right\}\\
& \times\exp\left\{\frac{F^2+H^2+2FHe^{-i\omega T}}{-4\hbar M\omega\sin^2\omega T}\right\}\\
= & \exp\left\{\frac{i}{2\hbar M\omega}\int_0^T\int_0^t\gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt\right\}\\
& \times\exp\left\{-\frac{F^2+2FH\cos\omega T+H^2}{4\hbar M\omega\sin^2\omega T}\right\}.
\end{aligned}
$$

But

$$\begin{aligned}
& F^2 + 2FH\cos\omega T + H^2 \\
= \;& F^2 + 2F\cos\omega T(G\sin\omega T - F\cos\omega T) + (G\sin\omega T - F\cos\omega T)^2 \\
= \;& F^2 + 2FG\cos\omega T\sin\omega T - 2F^2\cos^2\omega T + G^2\sin^2\omega T - 2FG\cos\omega T\sin\omega T + F^2\cos^2\omega T \\
= \;& F^2\sin^2\omega T + G^2\sin^2\omega T
\end{aligned}$$

so

$$\begin{aligned}
G_{00} &= \exp\left\{\frac{i}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt\right\} \\
&\quad \times\exp\left\{-\frac{F^2+G^2}{4\hbar M\omega}\right\} \\
&= \exp\left\{-\frac{1}{2\hbar M\omega}\left[\frac{1}{2}F^2 + \frac{1}{2}G^2 - i\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt\right]\right\}.
\end{aligned}$$

Meanwhile, Feynman and Hibbs give their answer in terms of the integral

$$\begin{aligned}
& \int_0^T\int_0^t \gamma(t)\gamma(s)e^{-i\omega(t-s)}\,ds\,dt \\
= \;& \int_0^T\int_0^t \gamma(t)\gamma(s)[\cos(\omega(t-s)) - i\sin(\omega(t-s))]\,ds\,dt \\
= \;& \int_0^T\int_0^t \gamma(t)\gamma(s)\cos(\omega(t-s))\,ds\,dt - i\int_0^T\int_0^t \gamma(t)\gamma(s)\sin(\omega(t-s))\,ds\,dt \\
= \;& \frac{1}{2}\int_0^T\int_0^T \gamma(t)\gamma(s)[\cos(\omega(t-s))]\,ds\,dt - i\int_0^T\int_0^t \gamma(t)\gamma(s)\sin(\omega(t-s))\,ds\,dt \\
= \;& \frac{1}{2}\int_0^T\int_0^T \gamma(t)\gamma(s)[\cos(\omega t)\cos(\omega s) + \sin(\omega t)\sin(\omega s)]\,ds\,dt \\
& \quad -i\int_0^T\int_0^t \gamma(t)\gamma(s)\sin(\omega(t-s))\,ds\,dt \\
= \;& \frac{1}{2}\int_0^T\int_0^T \gamma(t)\gamma(s)\cos(\omega t)\cos(\omega s)\,ds\,dt + \frac{1}{2}\int_0^T\int_0^T \gamma(t)\gamma(s)\sin(\omega t)\sin(\omega s)\,ds\,dt \\
& \quad -i\int_0^T\int_0^t \gamma(t)\gamma(s)\sin(\omega(t-s))\,ds\,dt \\
= \;& \frac{1}{2}\left(\int_0^T \gamma(t)\cos(\omega t)\,dt\right)^2 + \frac{1}{2}\left(\int_0^T \gamma(t)\sin(\omega t)\,dt\right)^2 \\
& \quad -i\int_0^T\int_0^t \gamma(t)\gamma(s)\sin(\omega(t-s))\,ds\,dt,
\end{aligned}$$

from which it is clear that

$$G_{00} = \exp\left\{-\frac{1}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)e^{-i\omega(t-s)}\,ds\,dt\right\}.$$

Part II: Transition amplitude from f to g. Evaluate $F(b,a)$ from equation (8-140) and the form of classical action equal to (3-66) given by equation (1) of my document "Kernel for the forced harmonic oscillator", obtaining equation (8-141).

$$\begin{aligned}
F(b,a) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g^*(x_b)K(x_b,T;x_a,0)f(x_a)\,dx_a\,dx_b \\
&= \left(\frac{M\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(M\omega/2\hbar)(x_b-b)^2}K(x_b,T;x_a,0)e^{-(M\omega/2\hbar)(x_a-a)^2}\,dx_a\,dx_b \\
&= \left(\frac{M\omega}{\pi\hbar}\right)^{1/2}\left(\frac{M\omega}{2\pi i\hbar\sin\omega T}\right)^{1/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(M\omega/2\hbar)(x_b-b)^2}e^{iS_{cl}/\hbar}e^{-(M\omega/2\hbar)(x_a-a)^2}\,dx_a\,dx_b \\
&= \left(\frac{M^2\omega^2}{2\pi^2 i\hbar^2\sin\omega T}\right)^{1/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iS_{cl}/\hbar}e^{-(M\omega/2\hbar)(x_a^2+x_b^2-2x_a a-2x_b b+a^2+b^2)}\,dx_a\,dx_b \\
&= \left(\frac{M^2\omega^2}{2\pi^2 i\hbar^2\sin\omega T}\right)^{1/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{\quad\}\,dx_a\,dx_b
\end{aligned}$$

where

$$\begin{aligned}
\{\quad\} = &+ \left[-\frac{M\omega}{2\hbar}+\frac{iM\omega}{2\hbar}\frac{\cos\omega T}{\sin\omega T}\right](x_a^2+x_b^2) \\
&- \frac{iM\omega}{\hbar\sin\omega T}x_a x_b \\
&+ \left[\frac{M\omega}{\hbar}b+\frac{i}{\hbar\sin\omega T}\int_0^T\gamma(t)\sin\omega t\,dt\right]x_b \\
&+ \left[\frac{M\omega}{\hbar}a+\frac{i}{\hbar\sin\omega T}\int_0^T\gamma(t)\sin\omega(T-t)\,dt\right]x_a \\
&- \frac{M\omega}{2\hbar}(a^2+b^2)-\frac{i}{2\hbar M\omega\sin\omega T}\int_0^T\gamma(t)\sin\omega t\,dt\int_0^T\gamma(t)\sin\omega(T-t)\,dt \\
&+ \frac{i}{2\hbar M\omega}\int_0^T\int_0^t\gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt
\end{aligned}$$

Define F, G, and H exactly as in part I. Then

$$\begin{aligned}
\{\quad\} = &+ \frac{iM\omega}{2\hbar\sin\omega T}e^{i\omega T}(x_a^2+x_b^2) \\
&- \frac{iM\omega}{\hbar\sin\omega T}x_a x_b \\
&+ \left[\frac{M\omega}{\hbar}b+\frac{iF}{\hbar\sin\omega T}\right]x_b \\
&+ \left[\frac{M\omega}{\hbar}a+\frac{iH}{\hbar\sin\omega T}\right]x_a \\
&- \frac{M\omega}{2\hbar}(a^2+b^2)-\frac{iFH}{2\hbar M\omega\sin\omega T}+\frac{i}{2\hbar M\omega}\int_0^T\int_0^t\gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt \\
\equiv\quad & D(x_a^2+x_b^2)+Cx_a x_b+Bx_b+Ax_a \\
&- \frac{M\omega}{2\hbar}(a^2+b^2)-\frac{iFH}{2\hbar M\omega\sin\omega T}+\frac{i}{2\hbar M\omega}\int_0^T\int_0^t\gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt
\end{aligned}$$

and

$$
\begin{aligned}
F(b,a) &= \left(\frac{M^2\omega^2}{2\pi^2 i\hbar^2 \sin\omega T}\right)^{1/2} \\
&\quad \times \exp\left\{-\frac{M\omega}{2\hbar}(a^2+b^2) - \frac{iFH}{2\hbar M\omega\sin\omega T} + \frac{i}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt\right\} \\
&\quad \times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{D(x_a^2+x_b^2)+Cx_ax_b+Bx_b+Ax_a\}\,dx_a\,dx_b
\end{aligned}
$$

This last integral is of the same form as came up in part I, and it falls readily to exactly the same rotation trick:

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{D(x_a^2+x_b^2)+Cx_ax_b+Bx_b+Ax_a\}\,dx_a\,dx_b \\
&= \frac{2\pi}{\sqrt{4D^2-C^2}}\exp\left\{\frac{ABC-D(A^2+B^2)}{4D^2-C^2}\right\}
\end{aligned}
$$

But from our definitions

$$
\begin{aligned}
4D^2-C^2 &= -\frac{2iM^2\omega^2}{\hbar^2\sin\omega T}e^{i\omega T} \\
D(A^2+B^2) &= \frac{iM\omega}{2\hbar\sin\omega T}e^{i\omega T}\left[\left(\frac{M\omega}{\hbar}a+\frac{iH}{\hbar\sin\omega T}\right)^2+\left(\frac{M\omega}{\hbar}b+\frac{iF}{\hbar\sin\omega T}\right)^2\right] \\
&= \frac{iM\omega}{2\hbar\sin\omega T}e^{i\omega T}\left[\left(\frac{M\omega}{\hbar}\right)^2(a^2+b^2)+\frac{i2M\omega}{\hbar^2\sin\omega T}(aH+bF)-\frac{F^2+H^2}{\hbar^2\sin^2\omega T}\right] \\
ABC &= -\frac{iM\omega}{\hbar\sin\omega T}\left(\frac{M\omega}{\hbar}a+\frac{iH}{\hbar\sin\omega T}\right)\left(\frac{M\omega}{\hbar}b+\frac{iF}{\hbar\sin\omega T}\right) \\
&= -\frac{iM\omega}{\hbar\sin\omega T}\left[\left(\frac{M\omega}{\hbar}\right)^2ab+\frac{iM\omega}{\hbar^2\sin\omega T}(aF+bH)-\frac{FH}{\hbar^2\sin^2\omega T}\right] \\
ABC-D(A^2+B^2) &= -\frac{iM\omega}{2\hbar\sin\omega T}e^{i\omega T}\left[\left(\frac{M\omega}{\hbar}\right)^2(a^2+b^2+2e^{-i\omega T}ab)\right. \\
&\quad \left. +\frac{i2M\omega}{\hbar^2\sin\omega T}[e^{-i\omega T}(aF+bH)+aH+bF]-\frac{1}{\hbar^2\sin^2\omega T}(F^2+2e^{-i\omega T}FH+H^2)\right]
\end{aligned}
$$

Looking at the middle term above

$$
e^{-i\omega T}(aF+bH)+aH+bF = a(Fe^{-i\omega T}+H)+b(F+He^{-i\omega T}) = a(Fe^{-i\omega T}+H)+b(Fe^{-i\omega T}+H)^*e^{-i\omega T}
$$

but

$$
\begin{aligned}
(Fe^{-i\omega T}+H) &= F\cos\omega T - iF\sin\omega T + G\sin\omega T - F\cos\omega T \\
&= \left[\int_0^T \gamma(t)\cos\omega t\,dt - i\int_0^T \gamma(t)\sin\omega t\,dt\right]\sin\omega T \\
&= \sin\omega T\int_0^T \gamma(t)e^{-i\omega t}\,dt
\end{aligned}
$$

so

$$
\begin{aligned}
& \frac{i2M\omega}{\hbar^2 \sin\omega T}[e^{-i\omega T}(aF+bH)+aH+bF] \\
= & \frac{i2M\omega}{\hbar^2}\left(a\int_0^T \gamma(t)e^{-i\omega t}\,dt + b\int_0^T \gamma(t)e^{+i\omega t}\,dt\,e^{-i\omega T}\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{ABC-D(A^2+B^2)}{4D^2-C^2} = & \frac{\hbar}{4M\omega}\left[\left(\frac{M\omega}{\hbar}\right)^2(a^2+b^2+2e^{-i\omega T}ab)\right. \\
& +\frac{i2M\omega}{\hbar^2}\left(a\int_0^T \gamma(t)e^{-i\omega t}\,dt + b\int_0^T \gamma(t)e^{+i\omega t}\,dt\,e^{-i\omega T}\right) \\
& \left.-\frac{F^2+2e^{-i\omega T}FH+H^2}{\hbar^2\sin^2\omega T}\right] \\
= & \frac{M\omega}{4\hbar}(a^2+b^2+2e^{-i\omega T}ab) \\
& +\frac{i}{2\hbar}\left(a\int_0^T \gamma(t)e^{-i\omega t}\,dt + b\int_0^T \gamma(t)e^{+i\omega t}\,dt\,e^{-i\omega T}\right) \\
& -\frac{F^2+2e^{-i\omega T}FH+H^2}{4\hbar M\omega\sin^2\omega T}
\end{aligned}
$$

Following Feynman and Hibbs, define

$$
\beta = \frac{1}{\sqrt{2\hbar M\omega}}\int_0^T \gamma(t)e^{-i\omega t}\,dt
$$

so that

$$
\begin{aligned}
\frac{ABC-D(A^2+B^2)}{4D^2-C^2} = & \frac{M\omega}{4\hbar}(a^2+b^2+2e^{-i\omega T}ab) \\
& +i\sqrt{\frac{M\omega}{2\hbar}}\left(a\beta+b\beta^* e^{-i\omega T}\right) \\
& -\frac{F^2+2e^{-i\omega T}FH+H^2}{4\hbar M\omega\sin^2\omega T}
\end{aligned}
$$

and the spatial integral is

$$
\begin{aligned}
& \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\{D(x_a^2+x_b^2)+Cx_ax_b+Bx_b+Ax_a\}\,dx_a\,dx_b \\
= & 2\pi\left(\frac{\hbar^2\sin\omega T}{-2iM^2\omega^2}\right)^{1/2} e^{-i\omega T/2}\exp\left\{\frac{M\omega}{4\hbar}(a^2+b^2+2e^{-i\omega T}ab)\right. \\
& +i\sqrt{\frac{M\omega}{2\hbar}}\left(a\beta+b\beta^* e^{-i\omega T}\right) \\
& \left.-\frac{F^2+2e^{-i\omega T}FH+H^2}{4\hbar M\omega\sin^2\omega T}\right\}
\end{aligned}
$$

Thus

$$
\begin{aligned}
F(b,a) &= \left(\frac{M^2\omega^2}{2\pi^2 i\hbar^2 \sin\omega T}\right)^{1/2} \\
&\quad \times \exp\left\{-\frac{M\omega}{2\hbar}(a^2+b^2) - \frac{iFH}{2\hbar M\omega\sin\omega T} + \frac{i}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt\right\} \\
&\quad \times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{D(x_a^2+x_b^2) + Cx_a x_b + Bx_b + Ax_a\}\,dx_a\,dx_b \\
&= e^{-i\omega T/2}\exp\left\{-\frac{M\omega}{4\hbar}(a^2+b^2-2e^{-i\omega T}ab)\right. \\
&\quad + i\sqrt{\frac{M\omega}{2\hbar}}\left(a\beta + b\beta^* e^{-i\omega T}\right) \\
&\quad - \frac{iFH}{2\hbar M\omega\sin\omega T} + \frac{i}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)\sin\omega(t-s)\,ds\,dt \\
&\quad \left. - \frac{F^2 + 2e^{-i\omega T}FH + H^2}{4\hbar M\omega\sin^2\omega T}\right\}
\end{aligned}
$$

If $a=b=0$ this is nearly the same as the G_{00} that we have already obtained at great expense in blood and toil. We conclude that

$$
\begin{aligned}
F(b,a) &= e^{-i\omega T/2}\exp\left\{-\frac{M\omega}{4\hbar}(a^2+b^2-2e^{-i\omega T}ab)\right. \\
&\quad + i\sqrt{\frac{M\omega}{2\hbar}}\left(a\beta + b\beta^* e^{-i\omega T}\right) \\
&\quad \left. - \frac{1}{2\hbar M\omega}\int_0^T\int_0^t \gamma(t)\gamma(s)e^{-i\omega(t-s)}\,ds\,dt\right\}
\end{aligned}
$$