Model Solutions to First Sample Exam

Note the following problem solving techniques:

- First step: draw a sketch.
- Select a strategy — the key idea to employ — before rushing in to make detailed derivations.
- When deriving, keep the goal in mind to avoid deriving endless numbers of equations that are true but don’t help you reach your goal.
- Check the final result for reasonableness.

1. X-rays
This problem asks about transformation of energy from one reference frame to another. The proper tool is the Lorentz transformation.

In laboratory frame:

\[ \text{photon} \quad E_g \]

All four-vectors, including \([ct, \vec{r}]\) and \([E/c, \vec{p}]\), have components that transform following the same pattern: for example, the Lorentz transformation for energy-momentum is

\[
\begin{align*}
E'/c &= \frac{E/c - (V/c)p_x}{\sqrt{1 - (V/c)^2}} \\
p'_x &= \frac{p_x - (V/c)(E/c)}{\sqrt{1 - (V/c)^2}} \\
p'_y &= p_y \\
p'_z &= p_z
\end{align*}
\]

For a photon, \(p_x = E/c\) so the first equation is

\[
\begin{align*}
E'/c &= \frac{(E/c)(1 - (V/c))}{\sqrt{1 - (V/c)^2}} = (E/c)\sqrt{1 - V/c \over 1 + V/c}.
\end{align*}
\]

In our case, \(E' = E/2 = 2.34\) KeV. It makes sense that the photon energy should be less in a frame running after the photon.
2. Two photons
We are given information about energies and asked about total masses, so it makes sense to use the invariant

\[(E_{\text{total}})^2 - (P_{\text{total}}c)^2 = (M_{\text{total}}c^2)^2.\]

a. Photon one going east, photon two going west:

\[
(M_{\text{total}}c^2)^2 = (E_1 + E_2)^2 - (p_1c - p_2c)^2
= E_1^2 + 2E_1E_2 + E_2^2 - [(p_1c)^2 - 2p_1cp_2c + (p_2c)^2]
= 2E_1E_2 - (2p_1cp_2c)
= 2E_1E_2 + 2E_1E_2
= 4E_1E_2
\]

\[
M_{\text{total}} = 2\sqrt{E_1E_2}/c^2
\]

[Where we’ve used the fact that for a photon — or any massless particle — \(E = pc\).]

b. Both photons going east:

\[
(M_{\text{total}}c^2)^2 = (E_1 + E_2)^2 - (p_1c + p_2c)^2
= E_1^2 + 2E_1E_2 + E_2^2 - [(p_1c)^2 + 2p_1cp_2c + (p_2c)^2]
= 2E_1E_2 - (2p_1cp_2c)
= 2E_1E_2 - 2E_1E_2
= 0
\]

\[
M_{\text{total}} = 0
\]
3. Photon absorption

In laboratory frame:

before:
- Photon
  - $E_{\gamma}$
- Ground-state atom
  - $m_g$

after:
- Excited atom
  - $m_e$

We are given information about energies and masses, and asked to find about mass. Thus it makes sense to use the conserved invariant

\[
(m_e c^2)^2 = (E_{\text{total}}^\text{initial})^2 - (p_{\text{total}}^\text{initial} c)^2 \\
= (E_{\gamma} + m_g c^2)^2 - (p_{\gamma} c)^2 \\
= E_{\gamma}^2 + 2E_{\gamma} m_g c^2 + (m_g c^2)^2 - (p_{\gamma} c)^2 \\
= 2E_{\gamma} m_g c^2 + (m_g c^2)^2 \\
= (m_g c^2)^2 (1 + 2E_{\gamma} / m_g c^2) \\
\]

\[
m_e = m_g \sqrt{1 + 2E_{\gamma} / m_g c^2}.
\]

As expected, $m_e$ is greater than $m_g$. 
4. Nuclear decay

In this case, we are asked about velocity as well as energy and mass, so the conserved invariant is not the most appropriate tool. Since the question asks solely about the lab-frame situation, we use conservation of energy and momentum in the laboratory frame.

In laboratory frame:

![Diagram](image)

Conserve energy:

\[ m_e c^2 = E_\gamma + \frac{m_g c^2}{\sqrt{1 - (v/c)^2}}. \] (1)

Conserve momentum:

\[ 0 = -p_\gamma + \frac{m_g v}{\sqrt{1 - (v/c)^2}}. \] (2)

The objective is to find the mass difference in terms of \( E_\gamma \) and \( v/c \). We accomplish this by finding the two masses individually in terms of \( E_\gamma \) and \( v/c \), then subtracting. The very first step is to eliminate \( p_\gamma \). From equation (2),

\[ E_\gamma = p_\gamma c = \left[ \frac{m_g v}{\sqrt{1 - (v/c)^2}} \right] c = \frac{v}{c} \left[ \frac{m_g c^2}{\sqrt{1 - (v/c)^2}} \right]. \] (3)

Solve for \( m_g c^2 \):

\[ m_g c^2 = E_\gamma \frac{\sqrt{1 - (v/c)^2}}{v/c}. \] (4)

Meanwhile, plug equation (3) into equation (1):

\[ m_e c^2 = E_\gamma + \frac{E_\gamma^2}{v/c} = E_\gamma \left[ 1 + \frac{1}{v/c} \right]. \] (5)

The difference of the two above equations is

\[ m_e c^2 - m_g c^2 = E_\gamma \left[ 1 + \frac{1}{v/c} - \frac{\sqrt{1 - (v/c)^2}}{v/c} \right]. \] (6)

Using the approximation \( \sqrt{1 - \epsilon} \approx 1 - \frac{1}{2} \epsilon \) gives

\[ m_e - m_g \approx \frac{E_\gamma}{c^2} \left[ 1 + \frac{1}{2} (v/c) \right]. \] (7)

As expected, \( m_e \) is greater than \( m_g \).
5. Nuclear fission

Since the question asks solely about the lab-frame situation, we use conservation of energy and momentum in the laboratory frame.

a. The initial lab-frame momentum was zero, so the final lab-frame momentum must be zero. If daughter A when straight north, then daughter B must go straight south.

b. Momentum conservation:

\[ 0 = p_A + p_B. \]  

I’ll use the symbol \( p = |p_A| = |p_B| \). Energy conservation:

\[ m_Nc^2 = E_A + E_B. \]  

Invariant for A and for B:

\[ \begin{align*}
E_A^2 - (pc)^2 &= (m_Ac^2)^2 \\
E_B^2 - (pc)^2 &= (m_Bc^2)^2.
\end{align*} \]  

Equations (9), (10), and (11) constitute three equations for the three unknowns \( E_A, E_B, \) and \( p \). We could solve for all three, but the question asks only for a solution for \( E_B \).

[Note that there are many other equations that we could have written down: We could have written down the formulas for \( E \) and \( p \) in terms of \( m \) and \( v \), we could have written down the equation for total mass, we could have written down the formula for \( v \) in terms of \( E \) and \( p \). But we didn’t write any of these down because we weren’t asked to find anything about the velocities or about the total mass. Life is hard enough as it is: Don’t distract yourself by writing down formulas for things you don’t need to find.]

Now I’m done all the physics and it’s just down to doing algebra. Because I get tired of always writing “\( m_Nc^2 \)”, I’ll just call it “\( N \)”. Similarly for \( A \) and \( B \). This will cut down on writing, but will also remove any temptation to break apart the natural combination of \( m_Nc^2 \).

First, eliminate \( p \) by subtracting equation (11) from (10):

\[ E_A^2 - E_B^2 = A^2 - B^2. \]

Second, substitute \( E_A = N - E_B \) (equation 9):

\[ \begin{align*}
(N - E_B)^2 - E_B^2 &= A^2 - B^2 \\
(N^2 - 2NE_B + E_B^2) - E_B^2 &= A^2 - B^2 \\
N^2 - 2NE_B &= A^2 - B^2 \\
E_B &= \frac{N^2 - A^2 + B^2}{2N}.
\end{align*} \]

Restore the meanings for the temporary symbols \( N, A, \) and \( B \) to produce the final result

\[ E_B = \frac{m_N^2 - m_A^2 + m_B^2}{2m_N}c^2. \]