Problem 2.4. Relativistic energy and momentum, I

A particle has relativistic energy equal to three times its rest energy. Find its resulting speed and momentum.

Answer:

\[ E = \frac{mc^2}{\sqrt{1 - (v/c)^2}} = 3mc^2, \]

So

\[ \sqrt{1 - (v/c)^2} = \frac{1}{3} \]
\[ 1 - (v/c)^2 = \frac{1}{9} \]
\[ (v/c)^2 = \frac{8}{9} \]
\[ v = \frac{\sqrt{8}}{3} c = 0.943 c. \]

The momentum is

\[ p = \frac{mv}{\sqrt{1 - (v/c)^2}} = 3mv = \sqrt{8} mc = 2.83 mc. \]

How do these results change if the total energy is six times its rest energy? Answer:

\[ E = \frac{mc^2}{\sqrt{1 - (v/c)^2}} = 6mc^2, \]

So

\[ \sqrt{1 - (v/c)^2} = \frac{1}{6} \]
\[ 1 - (v/c)^2 = \frac{1}{36} \]
\[ (v/c)^2 = \frac{35}{36} \]
\[ v = \frac{\sqrt{35}}{6} c = 0.986 c. \]

The momentum is

\[ p = \frac{mv}{\sqrt{1 - (v/c)^2}} = 6mv = \sqrt{35} mc = 5.92 mc. \]

Thus the total energy doubles, the momentum more than doubles, but the velocity increases just a bit.
Problem 4.1. Sticky particles

Before: 5 kg $\rightarrow$ (12/13)c 2 kg

After: $M$  $\rightarrow$ V

Conserve energy:

$$\frac{(5 \text{ kg})c^2}{\sqrt{1 - \left(\frac{12}{13}\right)^2}} + (2 \text{ kg})c^2 = \frac{Mc^2}{\sqrt{1 - (V/c)^2}}$$

$$\frac{(5 \text{ kg})c^2}{5/13} + (2 \text{ kg})c^2 = \frac{Mc^2}{\sqrt{1 - (V/c)^2}}$$

$$15 \text{ kg} = \frac{Mc^2}{\sqrt{1 - (V/c)^2}}$$

Conserve momentum:

$$\frac{(5 \text{ kg})(12/13)c}{\sqrt{1 - (12/13)^2}} + (2 \text{ kg})0 = \frac{MV}{\sqrt{1 - (V/c)^2}}$$

$$\frac{(5 \text{ kg})(12/13)c}{5/13} = \frac{MV}{\sqrt{1 - (V/c)^2}}$$

$$V = \frac{4}{5}c.$$  

(This speed is, of course, less than the incoming speed $\frac{12}{13}c$.)

Plug this value back into equation (1) to find

$$(12 \text{ kg})c = (15 \text{ kg})V$$

whence

$$V = \frac{4}{5}c.$$
So we have a 5 kg object sticking to a 2 kg object to form a composite of mass 9 kg, not 7 kg. Energy is conserved, but “sum of masses of constituents” is not!

**Problem 4.3: Sticky particles and the classical limit**

Initial: ball of mass $m$ and speed $v$ plus ball of mass $m$ and speed 0.

Final: ball of mass $M$ and speed $V$.


b. Relativistically — conserve momentum:

$$\frac{mv}{\sqrt{1-(v/c)^2}} = \frac{MV}{\sqrt{1-(V/c)^2}}.$$  (3)

Conserve energy:

$$\frac{mc^2}{\sqrt{1-(v/c)^2}} + mc^2 = \frac{Mc^2}{\sqrt{1-(V/c)^2}}.$$  (4)

Divide momentum equation by energy equation:

$$\frac{v/\sqrt{1-(v/c)^2}}{1/\sqrt{1-(v/c)^2} + 1} = V$$

or

$$V = \frac{v}{1 + \sqrt{1-(v/c)^2}}.$$

c. Classical limit: For $v \ll c$, $v/c \to 0$ and $V \to v/2$.

d. For any given $v$, the correct relativistic $V$ is always larger than the classical $V = v/2$. (In particular, as $v \to c$ then the correct result is $V \to c$ not $V \to c/2$.) [Note that all results obtained so far are independent of $m$.]

e. You can solve this through the algebra of solving equations (3) and (4) simultaneously, or you can use the conserved invariant:

$$\left(\frac{mc^2}{\sqrt{1-(v/c)^2}} + mc^2\right)^2 - \left(\frac{mvc}{\sqrt{1-(v/c)^2}}\right)^2 = (Mc^2)^2$$

$$\left(\frac{1}{\sqrt{1-(v/c)^2} + 1}\right)^2 - \left(\frac{v/c}{\sqrt{1-(v/c)^2}}\right)^2 = \left(\frac{M}{m}\right)^2$$

$$\frac{1 + 2\sqrt{1-(v/c)^2} + [1-(v/c)^2] - (v/c)^2}{1-(v/c)^2} = \left(\frac{M}{m}\right)^2$$

$$2\left(\frac{1-(v/c)^2 + \sqrt{1-(v/c)^2}}{1-(v/c)^2}\right) = \left(\frac{M}{m}\right)^2$$

$$m\sqrt{\frac{1}{1 + \sqrt{1-(v/c)^2}}} = M$$

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f. In the non-relativistic limit, $M \to 2m$.

g. In all cases, $M \geq 2m$.

**Problem 5.4: Sticky particles, II**

The quantity $E^2 - (pc)^2$ is invariant between frames and conserved through time. Initial values, in lab frame:

\[
E = \frac{m_1 c^2}{\sqrt{1 - (v/c)^2}} + m_2 c^2 = \frac{(5 \text{ kg}) c^2}{5/13} + (2 \text{ kg}) c^2 = (15 \text{ kg}) c^2 \\
\frac{p}{\sqrt{1 - (v/c)^2}} = \frac{m_1 v}{5/13} = (12 \text{ kg}) c \\
E^2 - (pc)^2 = (15 \text{ kg})^2 c^4 - (12 \text{ kg})^2 c^4 = (9 \text{ kg})^2 c^4
\]

Final values, in the lump’s frame:

\[
E = M c^2 \\
p = 0 \\
E^2 - (pc)^2 = M^2 c^4
\]

Using the conserved invariant, $M = 9 \text{ kg}$. Then, in the lab frame,

\[
\frac{V}{c} = \frac{pc}{E} = \frac{(12 \text{ kg}) c^2}{(15 \text{ kg}) c^2} = \frac{4}{5}.
\]