Quantum Mechanics 2023

Model Solutions for First Exam

1. Two analyzer loops

paths blocked	input state	path taken through # 1	intermediate state	path taken through # 2	output state	probability of input → output
none	$\lvert z-\rangle$	"both"	$\lvert z-\rangle$	b	$\lvert z-\rangle$	100%
2a	$\lvert z-\rangle$	"both"	$\lvert z-\rangle$	b	$\lvert z-\rangle$	100%
2b	$\lvert z-\rangle$	"both"	$\lvert z-\rangle$	100% blocked at b	none	0%
1a	$\lvert z-\rangle$	50% blocked at a 50% pass through b	$\lvert x-\rangle$	"both"	$\lvert x-\rangle$	50%
1b	$\lvert z-\rangle$	50% pass through a 50% blocked at b	$\lvert x+\rangle$	"both"	$\lvert x+\rangle$	50%
1b and 2a	$\lvert z-\rangle$	50% pass through a 50% blocked at b	$\lvert x+\rangle$	25% blocked at a 25% pass through b	$\lvert z-\rangle$	25%
1a and 2b	$\lvert z-\rangle$	50% blocked at a 50% pass through b	$\lvert x-\rangle$	25% pass through a 25% blocked at b	$\lvert z+\rangle$	25%

2. Orthonormality

Bases $\{|a_n\rangle\}$ and $\{|b_n\rangle\}$ are related through

$$\begin{aligned}
|b_1\rangle &= \cos\phi\,|a_1\rangle + \sin\phi\,|a_2\rangle \\
|b_2\rangle &= -\sin\phi\,|a_1\rangle + \cos\phi\,|a_2\rangle,
\end{aligned}$$

Show that if $\{|a_n\rangle\}$ is orthonormal then $\{|b_n\rangle\}$ is too.

$$\begin{aligned}
\langle b_1|b_1\rangle &= \Big\langle \Big[\cos\phi\,\langle a_1| + \sin\phi\,\langle a_2|\Big] \,\Big|\, \Big[\cos\phi\,|a_1\rangle + \sin\phi\,|a_2\rangle\Big]\Big\rangle \\
&= (\cos\phi)^2\,\langle a_1|a_1\rangle + \cos\phi\cdot\sin\phi\,\langle a_1|a_2\rangle + \sin\phi\cdot\cos\phi\,\langle a_2|a_1\rangle + (\sin\phi)^2\,\langle a_2|a_2\rangle \\
&= (\cos\phi)^2 + (\sin\phi)^2 = 1 \\
\langle b_1|b_2\rangle &= \Big\langle \Big[\cos\phi\,\langle a_1| + \sin\phi\,\langle a_2|\Big] \,\Big|\, \Big[-\sin\phi\,|a_1\rangle + \cos\phi\,|a_2\rangle\Big]\Big\rangle \\
&= -\cos\phi\cdot\sin\phi\,\langle a_1|a_1\rangle + (\cos\phi)^2\,\langle a_1|a_2\rangle - (\sin\phi)^2\,\langle a_2|a_1\rangle + \sin\phi\cdot\cos\phi\,\langle a_2|a_2\rangle \\
&= -\cos\phi\cdot\sin\phi + \sin\phi\cdot\cos\phi = 0 \\
\langle b_2|b_2\rangle &= \Big\langle \Big[-\sin\phi\,\langle a_1| + \cos\phi\,\langle a_2|\Big] \,\Big|\, \Big[-\sin\phi\,|a_1\rangle + \cos\phi\,|a_2\rangle\Big]\Big\rangle \\
&= (\sin\phi)^2\,\langle a_1|a_1\rangle - \sin\phi\cdot\cos\phi\,\langle a_1|a_2\rangle - \cos\phi\cdot\sin\phi\,\langle a_2|a_1\rangle + (\cos\phi)^2\,\langle a_2|a_2\rangle \\
&= (\sin\phi)^2 + (\cos\phi)^2 = 1.
\end{aligned}$$

3. Change of basis

a. The set $\{|1'\rangle, |2'\rangle\}$ spans because $\{|1\rangle, |2\rangle\}$ spans and $|1\rangle = |1'\rangle$; $|2\rangle = e^{i\phi}|2'\rangle$.
$|1'\rangle$ and $|2'\rangle$ are orthogonal because $\langle 1'|2'\rangle = e^{-i\phi}\langle 1|2\rangle = 0$.
Each element is normalized because $\langle 1'|1'\rangle = \langle 1|1\rangle = 1$; $\langle 2'|2'\rangle = e^{-i\phi}\langle 2'|2\rangle = e^{-i\phi}e^{i\phi}\langle 2|2\rangle = 1$.

b. The desired matrix is
$$\begin{pmatrix} \langle 1'|\hat{H}|1'\rangle & \langle 1'|\hat{H}|2'\rangle \\ \langle 2'|\hat{H}|1'\rangle & \langle 2'|\hat{H}|2'\rangle \end{pmatrix}.$$

But

$$\begin{aligned}
\langle 1'|\hat{H}|1'\rangle &= \langle 1|\hat{H}|1\rangle = a \\
\langle 1'|\hat{H}|2'\rangle &= e^{-i\phi}\langle 1|\hat{H}|2\rangle = e^{-i\phi}(ce^{i\phi}) = c \\
\langle 2'|\hat{H}|1'\rangle &= e^{i\phi}\langle 2|\hat{H}|1\rangle = e^{i\phi}(ce^{-i\phi}) = c \\
\langle 2'|\hat{H}|2'\rangle &= e^{i\phi}e^{-i\phi}\langle 2|\hat{H}|2\rangle = b
\end{aligned}$$

so the desired matrix is
$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

4. Matrix algebra

a. From the problem assignment, the matrix is
$$\begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}.$$

If this matrix is to represent a Hamiltonian, the diagonal elements must be real. That is,

$$\Im m\{z_0 + z_3\} = \Im m\{z_0 - z_3\} = 0$$

or
$$\Im m\{z_0\} + \Im m\{z_3\} = 0 \quad \text{while} \quad \Im m\{z_0\} - \Im m\{z_3\} = 0.$$

Solving these two equations simultaneously gives

$$\Im m\{z_0\} = 0 \quad \text{and} \quad \Im m\{z_3\} = 0,$$

that is z_0 and z_3 must be pure real.

In addition, if this matrix is to represent a Hamiltonian, the subdiagonal is the complex conjugate of the superdiagonal:
$$z_1 + iz_2 = (z_1 - iz_2)^* = z_1^* + iz_2^*$$

whence
$$z_1 - z_1^* = i(z_2^* - z_2) \quad \text{or} \quad 2i\Im m\{z_1\} = i(-2i\Im m\{z_2\}).$$

In this equation, the left-hand side is pure imaginary, while the right-hand side is pure real. Hence both sides must vanish,
$$\Im m\{z_1\} = \Im m\{z_2\} = 0,$$

that is z_1 and z_2 must be pure real.

b. Problem 3(b) shows that with a suitable choice of basis, the off-diagonal element of a 2×2 Hamiltonian can always be made real. Using this choice we will thus have $z_2 = 0$.