Quantum Mechanics 2023

Model Solutions for Second Exam

1. Commutator

Apply

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = c\hat{B}\hat{A}$$

to the ket $|a\rangle$ that has $\hat{A}|a\rangle = a|a\rangle$. We find

$$\begin{aligned}
\hat{A}\hat{B}|a\rangle - \hat{B}\hat{A}|a\rangle &= c\hat{B}\hat{A}|a\rangle \\
\hat{A}\hat{B}|a\rangle - \hat{B}(a|a\rangle) &= c\hat{B}(a|a\rangle) \\
\hat{A}(\hat{B}|a\rangle) - a(\hat{B}|a\rangle) &= ca(\hat{B}|a\rangle) \\
\hat{A}(\hat{B}|a\rangle) &= a(\hat{B}|a\rangle) + ca(\hat{B}|a\rangle) \\
\hat{A}(\hat{B}|a\rangle) &= a(1+c)(\hat{B}|a\rangle)
\end{aligned}$$

So $\hat{B}|a\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue $a(1+c)$.

2. Real wavefunction

The operator $\hat{p}$ is Hermitian, so

$$\langle\psi|\hat{p}|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle^*.$$

That is, $\langle\psi|\hat{p}|\psi\rangle$ is pure real.

But the position-basis expression for $\langle\psi|\hat{p}|\psi\rangle$ is

$$\langle\psi|\hat{p}|\psi\rangle = \int_{-\infty}^{+\infty} \psi^*(x) \left(-i\hbar\frac{\partial}{\partial x}\right) \psi(x)\, dx.$$

Because $\psi(x)$ is pure real,

$$\langle\psi|\hat{p}|\psi\rangle = -i\hbar \int_{-\infty}^{+\infty} \psi(x) \frac{\partial\psi(x)}{\partial x}\, dx$$

is pure imaginary.

There is only one number that is both pure real and pure imaginary, and that number is zero. Thus

$$\langle\hat{p}\rangle = 0.$$

3. Scaled variables for the infinite square well

For the infinite square well,

$$V(x) = \begin{cases} \infty & \text{for } x \le 0 \\ 0 & \text{for } 0 < x < L \\ \infty & \text{for } L \le x \end{cases}$$

and the Schrödinger equation for our ambivating particle is

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi\right].$$

The solutions can depend upon only these three parameters:

parameter	dimensions
m	$[M]$
L	$[L]$
$\hbar$	$[ML^2/T]$

The characteristic length ξ is of course

$$\xi = L.$$

The characteristic time τ can only be

$$\tau = mL^2/\hbar.$$

Define the scaled time $\tilde{t} = t/\tau$, the scaled length $\tilde{x} = x/\xi$, and the scaled potential energy function

$$\tilde{V}(\tilde{x}) = \begin{cases} \infty & \text{for } \tilde{x} \le 0 \\ 0 & \text{for } 0 < \tilde{x} < 1 \\ \infty & \text{for } 1 \le \tilde{x} \end{cases}$$

In terms of these variables

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} = \frac{\partial \psi}{\partial \tilde{t}}\frac{1}{\tau} = \frac{\partial \psi}{\partial \tilde{t}}\frac{\hbar}{mL^2},$$

while

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial \tilde{x}^2}\frac{1}{\xi^2} = \frac{\partial^2 \psi}{\partial \tilde{x}^2}\frac{1}{L^2}.$$

So the Schrödinger equation is

$$\frac{\partial \psi}{\partial \tilde{t}}\frac{\hbar}{mL^2} = -\frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial \tilde{x}^2}\frac{1}{L^2} + \tilde{V}(\tilde{x})\psi\right]$$

or

$$\frac{\partial \psi}{\partial \tilde{t}} = -i\left[-\frac{1}{2}\frac{\partial^2 \psi}{\partial \tilde{x}^2} + \tilde{V}(\tilde{x})\psi\right].$$

4. Time evolution for the infinite square well.

The wavefunction evolves in time according to

$$\psi(x,t) = \sum_{n=1}^{\infty} C_n e^{-(i/\hbar)E_n t}\, \eta_n(x).$$

Suppose there were a "revival time" T_{rev} such that

$$e^{-(i/\hbar)E_n T_{\text{rev}}} = 1 \qquad \text{for } n = 1, 2, 3, \ldots.$$

The wavefunction $\psi(x,t)$ at time $t = T_{\text{rev}}$ would be exactly equal to the initial wavefunction $\psi_0(x)$!

Does such a revival time exist? Because $e^{-i\,2\pi\,\text{integer}} = 1$ for any integer, the revival conditions above are equivalent to

$$(1/\hbar)E_n T_{\text{rev}} = 2\pi(\text{an integer}) \qquad \text{for } n = 1, 2, 3, \ldots.$$

Combined with the energy eigenvalues, these conditions are

$$n^2 \frac{\pi\hbar}{4mL^2} T_{\text{rev}} = (\text{an integer}) \qquad \text{for } n = 1, 2, 3, \ldots.$$

And it's clear that yes, there is a time T_{rev} that satisfies this infinite number of conditions. The smallest such time is

$$T_{\text{rev}} = \frac{4mL^2}{\pi\hbar}.$$