Exercises on Formalism

Interpretation of amplitude squared as a probability

From the Schwarz inequality:

$$\begin{aligned} |\langle a_n|\psi\rangle| &\leq \sqrt{\langle a_n|a_n\rangle}\sqrt{\langle\psi|\psi\rangle} \\ |\langle a_n|\psi\rangle|^2 &\leq \langle a_n|a_n\rangle\langle\psi|\psi\rangle. \end{aligned}$$

But $\langle a_n|a_n\rangle = 1$ by orthonormality, and $\langle\psi|\psi\rangle = 1$ by normalization of states. Furthermore, any complex number has non-negative square modulus, so

$$0 \leq |\langle a_n|\psi\rangle|^2 \leq 1.$$

⟦*Grading:* 2 points for mentioning "Schwarz inequality"; 6 points for using it; 2 points for pointing out that "any complex number has non-negative square modulus".⟧

Mean value

$$\begin{aligned} |\psi\rangle &= \sum_n |a_n\rangle\langle a_n|\psi\rangle \\ \hat{A}|\psi\rangle &= \sum_m (\hat{A}|a_m\rangle)\langle a_m|\psi\rangle \\ &= \sum_m a_m|a_m\rangle\langle a_m|\psi\rangle \end{aligned}$$

So

$$\begin{aligned} \langle\psi|\hat{A}|\psi\rangle &= \left[\sum_n \langle\psi|a_n\rangle\langle a_n|\right]\left[\sum_m a_m|a_m\rangle\langle a_m|\psi\rangle\right] \\ &= \sum_n\sum_m \langle\psi|a_n\rangle a_m\langle a_n|a_m\rangle\langle a_m|\psi\rangle \\ &= \sum_n\sum_m \langle\psi|a_n\rangle a_m\delta_{n,m}\langle a_m|\psi\rangle \\ &= \sum_n \langle\psi|a_n\rangle a_n\langle a_n|\psi\rangle \\ &= \sum_n a_n|\langle a_n|\psi\rangle|^2 \\ &= \langle\hat{A}\rangle. \end{aligned}$$

Measurement example

Eigenbases $\{|a_n\rangle\}$ and $\{|b_n\rangle\}$ are related through

$$\begin{aligned} |b_1\rangle &= \tfrac{4}{5}|a_1\rangle + \tfrac{3}{5}|a_2\rangle \\ |b_2\rangle &= -\tfrac{3}{5}|a_1\rangle + \tfrac{4}{5}|a_2\rangle \end{aligned}$$

a. Show that if $\{|a_n\rangle\}$ is orthonormal then $\{|b_n\rangle\}$ is too.

$$\begin{aligned} \langle b_1|b_1\rangle &= \left\langle \left[\tfrac{4}{5}\langle a_1| + \tfrac{3}{5}\langle a_2|\right] \mid \left[\tfrac{4}{5}|a_1\rangle + \tfrac{3}{5}|a_2\rangle\right]\right\rangle \\ &= \left(\tfrac{4}{5}\right)^2 \langle a_1|a_1\rangle + \tfrac{4}{5}\cdot\tfrac{3}{5}\langle a_1|a_2\rangle + \tfrac{3}{5}\cdot\tfrac{4}{5}\langle a_2|a_1\rangle + \left(\tfrac{3}{5}\right)^2 \langle a_2|a_2\rangle \\ &= \left(\tfrac{4}{5}\right)^2 + \left(\tfrac{3}{5}\right)^2 = 1 \\ \langle b_1|b_2\rangle &= \left\langle \left[\tfrac{4}{5}\langle a_1| + \tfrac{3}{5}\langle a_2|\right] \mid \left[-\tfrac{3}{5}|a_1\rangle + \tfrac{4}{5}|a_2\rangle\right]\right\rangle \\ &= -\tfrac{4}{5}\cdot\tfrac{3}{5}\langle a_1|a_1\rangle + \left(\tfrac{4}{5}\right)^2 \langle a_1|a_2\rangle - \left(\tfrac{3}{5}\right)^2 \langle a_2|a_1\rangle + \tfrac{3}{5}\cdot\tfrac{4}{5}\langle a_2|a_2\rangle \\ &= -\tfrac{4}{5}\cdot\tfrac{3}{5} + \tfrac{3}{5}\cdot\tfrac{4}{5} = 0 \\ \langle b_2|b_2\rangle &= \left\langle \left[-\tfrac{3}{5}\langle a_1| + \tfrac{4}{5}\langle a_2|\right] \mid \left[-\tfrac{3}{5}|a_1\rangle + \tfrac{4}{5}|a_2\rangle\right]\right\rangle \\ &= \left(-\tfrac{3}{5}\right)^2 \langle a_1|a_1\rangle - \tfrac{3}{5}\cdot\tfrac{4}{5}\langle a_1|a_2\rangle - \tfrac{4}{5}\cdot\tfrac{3}{5}\langle a_2|a_1\rangle + \left(\tfrac{4}{5}\right)^2 \langle a_2|a_2\rangle \\ &= \left(\tfrac{3}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2 = 1. \end{aligned}$$

b. Find $\{|a_n\rangle\}$ in terms of $\{|b_n\rangle\}$.

Straightforward linear algebra gives

$$\begin{aligned} |a_1\rangle &= \tfrac{4}{5}|b_1\rangle - \tfrac{3}{5}|b_2\rangle \\ |a_2\rangle &= \tfrac{3}{5}|b_1\rangle + \tfrac{4}{5}|b_2\rangle \end{aligned}$$

c. Repeated measurements. $\hat{A}$ is measured, giving a_1. The system is now in state $|a_1\rangle = \tfrac{4}{5}|b_1\rangle - \tfrac{3}{5}|b_2\rangle$. Then $\hat{B}$ is measured.

Possibility I: Measurement of $\hat{B}$ results in b_1. This happens with probability $\left(\tfrac{4}{5}\right)^2$, and the system is now in state $|b_1\rangle = \tfrac{4}{5}|a_1\rangle + \tfrac{3}{5}|a_2\rangle$. So when $\hat{A}$ is measured again, the result is a_1 with probability $\left(\tfrac{4}{5}\right)^2$, the result is a_2 with probability $\left(\tfrac{3}{5}\right)^2$.

Possibility II: Measurement of $\hat{B}$ results in b_2. This happens with probability $\left(-\tfrac{3}{5}\right)^2$, and the system is now in state $|b_2\rangle = -\tfrac{3}{5}|a_1\rangle + \tfrac{4}{5}|a_2\rangle$. So when $\hat{A}$ is measured again, the result is a_1 with probability $\left(-\tfrac{3}{5}\right)^2$, the result is a_2 with probability $\left(\tfrac{4}{5}\right)^2$.

$$\begin{aligned} \text{probability of measuring } a_1 \text{ through possibility I} &= \left(\tfrac{4}{5}\right)^2\left(\tfrac{4}{5}\right)^2 = \tfrac{256}{625} \\ \text{probability of measuring } a_1 \text{ through possibility II} &= \left(-\tfrac{3}{5}\right)^2\left(-\tfrac{3}{5}\right)^2 = \tfrac{81}{625} \\ \text{total probability of measuring } a_1 &= \tfrac{337}{625} \end{aligned}$$

$$\begin{aligned}
\text{probability of measuring } a_2 \text{ through possibility I} &= \left(\tfrac{4}{5}\right)^2\left(\tfrac{3}{5}\right)^2 = \tfrac{144}{625}\\
\text{probability of measuring } a_2 \text{ through possibility II} &= \left(-\tfrac{3}{5}\right)^2\left(\tfrac{4}{5}\right)^2 = \tfrac{144}{625}\\
\text{total probability of measuring } a_2 &= \tfrac{288}{625}
\end{aligned}$$

The answers do indeed sum to 1. This is not proof that they're correct, but if they had summed to something other than 1, that would have been proof that they *weren't* correct!

Example of generalized indeterminancy relation

For this case $\Delta\hat{\mu}_z = 0$ and $\Delta\hat{\mu}_x = \mu_B/\sqrt{2}$.

Meanwhile, in the $\{|z+\rangle, |z-\rangle\}$ basis (see textbook, equation (3.13)),

$$|z+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \hat{\mu}_z \doteq \mu_B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \hat{\mu}_x \doteq \mu_B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So in this basis

$$\begin{aligned}
[\hat{\mu}_z, \hat{\mu}_x] &\doteq \mu_B^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \mu_B^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\\
&= \mu_B^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \mu_B^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\\
&= 2\mu_B^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\end{aligned}$$

whence

$$\begin{aligned}
\langle z + |[\hat{\mu}_z, \hat{\mu}_x]|z+\rangle &= 2\mu_B^2 \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\\
&= 2\mu_B^2 \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\\
&= 0.
\end{aligned}$$

So both sides of the generalized indeterminancy relation are zero, and sure enough

$$0 \leq 0.$$