## Expressions for SHO Ladder Operators

The lowering operator $\hat{a}$ acts upon energy eigenstate $|n\rangle$ as

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle.$$

Since we know how $\hat{a}$ acts upon every element of a basis, we know how it acts upon any state.

The outer product expression

$$\sum_{m=0}^{\infty} \sqrt{m}\,|m-1\rangle\langle m|$$

similarly takes in any energy state $|n\rangle$ and spits out $\sqrt{n}\,|n-1\rangle$. It must be the same operator.

The $m$-th row, $n$-th column matrix element (in the energy basis) is

$$a_{m,n} = \langle m|\hat{a}|n\rangle = \sqrt{n}\,\delta_{m,n-1}$$

so the matrix representation (in the energy basis) is

$$\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \\ 0 & 0 & 0 & 0 & 0 & \\ & & \vdots & & & \ddots \end{pmatrix}.$$

Now, as far as $\hat{a}^\dagger$ is concerned, because

$$(|p\rangle\langle q|)^\dagger = |q\rangle\langle p|,$$

we have

$$\hat{a}^\dagger = \sum_{m=0}^{\infty} \sqrt{m}\,|m\rangle\langle m-1| = \sum_{m=1}^{\infty} \sqrt{m}\,|m\rangle\langle m-1| = \sum_{m=0}^{\infty} \sqrt{m+1}\,|m+1\rangle\langle m|,$$

and a matrix representation (in the energy basis) of

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \\ \sqrt{1} & 0 & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \\ & & \vdots & & & \ddots \end{pmatrix}.$$

⟦*Grading:* There are many ways to approach this problem. Grade for a total of 10 points recognizing that there will be wide variation in approach.⟧