Photon Polarization

1. Classical description of polarized light

Recall that the intensity of a light beam is proportional to the square of its amplitude. That is, if a light beam is $E(z, t) = E_0 \cos(kz - \omega t)$, then its amplitude is proportional to $|E_0|^2$.

The figure makes it clear that as the $x$-polarized light passes through the polaroid sheet, the component $E_0 \sin \theta$ is erased. The $\theta$-polarized beam has amplitude $E_0 \cos \theta$, so the beam intensity is diminished from $I_0$ to $I_0 \cos^2 \theta$.

2. Quantal description of polarized light: Analyzers

$$|\langle x | \theta \rangle|^2 = \cos^2 \theta \quad |\langle x | \theta + 90^\circ \rangle|^2 = \sin^2 \theta$$

These analyzer (or “measurement”) experiments determine the magnitude of each probability amplitude.

The two states are complete because any incoming photon emerges from either the $\theta$ port or the $\theta + 90^\circ$ port.

The two states are orthogonal because when a $|\theta\rangle$ photon encounters a $\theta$-analyzer, it emerges from the $\theta$ port with probability one and from the $\theta + 90^\circ$ port with probability zero.
3. Interference

Form a $\theta$ analyzer loop by tilting the $x, y$ analyzer loop by the angle $\theta$.

Experiment 1: Slot a blocked.

- probability of passing from input to intermediate is $|\langle \theta + 90^\circ | x \rangle|^2 = \sin^2 \theta$
- state of intermediate photon is $|\theta + 90^\circ \rangle$
- probability of passing from intermediate to output is $|\langle y | \theta + 90^\circ \rangle|^2 = \cos^2 \theta$
- probability of passing from input to output is $\sin^2 \theta \cos^2 \theta$

Experiment 2: Slot b blocked.

- probability of passing from input to intermediate is $|\langle \theta | x \rangle|^2 = \cos^2 \theta$
- state of intermediate photon is $|\theta \rangle$
- probability of passing from intermediate to output is $|\langle y | \theta \rangle|^2 = \sin^2 \theta$
- probability of passing from input to output is $\cos^2 \theta \sin^2 \theta$

Experiment 3: Both slots open.

- probability of passing from input to intermediate is 1
- state of intermediate photon is $|x \rangle$
- probability of passing from intermediate to output is $|\langle y | x \rangle|^2 = 0$
- probability of passing from input to output is 0

And clearly,

$$0 \neq 2 \sin^2 \theta \cos^2 \theta$$

An equation representing experiment 3 is:

$$\langle y | \theta \rangle \langle \theta | x \rangle + \langle y | \theta + 90^\circ \rangle \langle \theta + 90^\circ | x \rangle = \langle y | x \rangle = 0$$

(1)
Problem 2 above gives us the magnitudes
\[ |\langle x|\theta \rangle| = |\cos \theta| \quad |\langle x|\theta + 90^\circ \rangle| = |\sin \theta|. \] (2)

And because \(|\langle x|\theta \rangle|^2 + |\langle y|\theta \rangle|^2 = 1; |\langle x|\theta + 90^\circ \rangle|^2 + |\langle y|\theta + 90^\circ \rangle|^2 = 1\), we have
\[ |\langle y|\theta \rangle| = |\sin \theta| \quad |\langle y|\theta + 90^\circ \rangle| = |\cos \theta|. \] (3)

The easiest way to satisfy equations (1), (2), and (3) simultaneously is to pick all the amplitudes real and one of them negative. The conventional choice is
\[
\begin{align*}
\langle x|\theta \rangle & = \cos \theta \\
\langle y|\theta \rangle & = \sin \theta \\
\langle x|\theta + 90^\circ \rangle & = -\sin \theta \\
\langle y|\theta + 90^\circ \rangle & = \cos \theta
\end{align*}
\]

[] A general point on determining probability amplitudes: Analyzer experiments give us the magnitudes through equations like (2) and (3), while interference experiments give us the phases through equations like (1).

4. Circular polarization

Can real values
\[
\langle R|\ell p \rangle = \pm 1/\sqrt{2} \quad \langle L|\ell p \rangle = \pm 1/\sqrt{2}
\]
satisfy
\[
\langle \theta|R \rangle \langle R|x \rangle + \langle \theta|L \rangle \langle L|x \rangle = \langle \theta|x \rangle = \cos \theta
\]

Certainly not! In any such attempt the left-hand side could take on only three possible values, namely 1, 0, or −1, and the the right-hand side \cos \theta certainly takes on values other than these! However, the complex amplitudes suggested in the question do work, because
\[
\begin{align*}
\langle \theta|R \rangle \langle R|x \rangle + \langle \theta|L \rangle \langle L|x \rangle & = (e^{i\theta}/\sqrt{2})(1/\sqrt{2}) + (e^{-i\theta}/\sqrt{2})(1/\sqrt{2}) \\
& = \frac{e^{i\theta} + e^{-i\theta}}{2} \\
& = \cos \theta.
\end{align*}
\]