

Model Solutions for Final Exam

1. Change of chemical potential with temperature

The master function associated with variables (T, p, N) is

$$dG = -S dT + V dp + \mu dN,$$

whence

$$\left. \frac{\partial G}{\partial T} \right)_{p,N} = -S.$$

But $G = \mu N$ so

$$\left. \frac{\partial \mu}{\partial T} \right)_{p,N} = -\frac{S}{N}.$$

2. Generalized equipartition theorem and the ultra-relativistic gas

a. The proof proceeds exactly as does the proof of the ordinary equipartition theorem up to equation (5.22):

$$\langle a|p|^n \rangle = -\frac{d}{d\beta} \ln \left[\int_{-\infty}^{+\infty} dp e^{-\beta a|p|^n} \right].$$

Because the integrand is even,

$$\langle a|p|^n \rangle = -\frac{d}{d\beta} \ln \left[2 \int_0^{+\infty} dp e^{-\beta a p^n} \right].$$

(This trick means that we can always consider p to be positive, and don't need to worry about the absolute value function, which is non-analytic at the origin.) Employ the substitution

$$u = (\beta a)^{1/n} p$$

to find

$$\begin{aligned} \langle a|p|^n \rangle &= -\frac{d}{d\beta} \ln \left[\frac{2}{(\beta a)^{1/n}} \int_0^{+\infty} du e^{-u^n} \right] \\ &= -\frac{d}{d\beta} \left\{ \ln \left[\frac{1}{\beta^{1/n}} \right] + \ln \left[\frac{2}{a^{1/n}} \int_0^{+\infty} du e^{-u^n} \right] \right\}. \end{aligned}$$

The term in square brackets on the bottom right is not a function of β and hence vanishes when the derivative is taken, leaving

$$\langle a|p|^n \rangle = -\frac{d}{d\beta} \ln \beta^{-1/n} = \frac{1}{n} \frac{d}{d\beta} \ln \beta = \frac{1}{n} \frac{1}{\beta}$$

or, finally,

$$\langle a|p|^n \rangle = \frac{1}{n} k_B T.$$

b. For the ultra-relativistic gas with N particles, the exponent is $n = 1$ and there are $3N$ terms in the Hamiltonian like pc . Thus the mean total energy is $3Nk_B T$ and the heat capacity $\partial E / \partial T$ is $3Nk_B \dots$ twice the classical result.

c. Crossover comes when $k_B T \approx mc^2$.

Note: An ideal gas consisting of particles will, at very high temperatures, cross over from the classical heat capacity of $\frac{3}{2}k_B N$ to the ultrarelativistic heat capacity of $3k_B N$. A real gas made of atoms, rather than particles, behaves differently: the atoms ionize (and the gas becomes a plasma) at temperatures much lower than this crossover temperature.

3. Fermion gas in two dimensions

a. For $d = 3$, the density of levels is $2 \frac{V}{(2\pi)^3}$; thus for $d = 2$, the density of levels is $2 \frac{A}{(2\pi)^2}$.

b. Using $\mathcal{E}_F = \frac{\hbar^2 k_F^2}{2m}$, the area of the “Fermi disk” is

$$\pi k_F^2 = \frac{2\pi m}{\hbar^2} \mathcal{E}_F.$$

The number of levels in the Fermi disk is

$$\frac{A}{2\pi^2} \times \frac{2\pi m}{\hbar^2} \mathcal{E}_F = N \quad \text{whence} \quad \mathcal{E}_F = \frac{\pi \hbar^2}{m} \frac{N}{A}.$$

c. At finite temperature

$$\begin{aligned} N &= \sum_r \langle n_r \rangle \\ &= \frac{A}{2\pi^2} \int d^2 k \frac{1}{e^{\beta(\mathcal{E}(k)-\mu)} + 1} \\ &= \frac{A}{2\pi^2} 2\pi \int_0^\infty k dk \frac{1}{e^{\beta(\mathcal{E}(k)-\mu)} + 1} \quad [\dots \text{use the substitution } \mathcal{E} = \hbar^2 k^2 / 2m \dots] \\ &= \frac{Am}{\pi \hbar^2} \int_0^\infty d\mathcal{E} \frac{1}{e^{\beta(\mathcal{E}-\mu)} + 1} \end{aligned}$$

so

$$\int_0^\infty d\mathcal{E} \frac{1}{e^{\beta(\mathcal{E}-\mu)} + 1} = \frac{\pi \hbar^2}{m} \frac{N}{A} = \mathcal{E}_F.$$

This integral is evaluated in Dwight, equation 569.1:

$$\begin{aligned} \int_0^\infty d\mathcal{E} \frac{1}{e^{\beta(\mathcal{E}-\mu)} + 1} &= \left[\mathcal{E} - \frac{1}{\beta} \ln(e^{\beta(\mathcal{E}-\mu)} + 1) \right]_0^\infty \\ &= \lim_{\mathcal{E} \rightarrow \infty} \left[\mathcal{E} - \frac{1}{\beta} [\beta(\mathcal{E}-\mu)] \right] + \frac{1}{\beta} \ln(e^{-\beta\mu} + 1) \\ &= \mu + \frac{1}{\beta} \ln(e^{-\beta\mu} + 1). \end{aligned}$$

Whence

$$\mathcal{E}_F = \mu + k_B T \ln(1 + e^{-\mu/k_B T}).$$

d. First note that $\ln(1 + e^{-\mu/k_B T})$ is bigger than zero and increases with increasing T . Now, at $T = 0$ we have $\mathcal{E}_F = \mu$. Remember that \mathcal{E}_F doesn't depend on T . So, for $T > 0$ we must have μ decreasing in order that the right-hand of the equation above still sums to \mathcal{E}_F .

4. Polymers

a. There are six possible walks of a single step: north, south, east, west, up, and down. Thus $N_1 = 6$. The same six possible steps may be appended to the tip of any walk, whence $N_n = 6N_{n-1}$. Thus $N_n = 6^n$. This formula even makes sense when $n = 0$, because there is one walk consisting of no steps.

b. Use the slick trick:

$$\Xi(\alpha) = \sum_x e^{-\alpha n(x)}$$

so

$$\langle n \rangle = \frac{\sum_x n(x) e^{-\alpha n(x)}}{\sum_x e^{-\alpha n(x)}} = -\frac{\partial \ln \Xi(\alpha)}{\partial \alpha}.$$

c. The partition function is

$$\begin{aligned} \Xi(\alpha) &= \sum_x e^{-\alpha n(x)} \\ &= \sum_{n=0}^{\infty} N_n e^{-\alpha n} \\ &= \sum_{n=0}^{\infty} 6^n e^{-\alpha n} \\ &= \sum_{n=0}^{\infty} (6e^{-\alpha})^n \\ &= \frac{1}{1 - 6e^{-\alpha}}, \end{aligned}$$

where the series converges when $6e^{-\alpha} < 1$, that is when $\alpha > \ln 6$. Thus

$$\begin{aligned} \langle n \rangle &= -\frac{\partial \ln \Xi(\alpha)}{\partial \alpha} \\ &= \frac{\partial \ln(1 - 6e^{-\alpha})}{\partial \alpha} \\ &= \frac{6e^{-\alpha}}{1 - 6e^{-\alpha}} \\ &= \frac{6}{e^{\alpha} - 6}. \end{aligned}$$

d. The control parameter α ranges from ∞ (associated with generally short polymers) to its minimum value $\ln 6$ (associated with generally long polymers).

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e.

$$\begin{aligned}\frac{\partial^2 \ln \Xi}{\partial \alpha^2} &= -\frac{\partial \langle n \rangle}{\partial \alpha} \\ &= -\frac{\partial}{\partial \alpha} \left(\frac{\sum_x n(x) e^{-\alpha n(x)}}{\sum_x e^{-\alpha n(x)}} \right) \\ &= -\frac{(\sum_x e^{-\alpha n(x)}) (\sum_x -n^2(x) e^{-\alpha n(x)}) - (\sum_x -n(x) e^{-\alpha n(x)}) (\sum_x n(x) e^{-\alpha n(x)})}{(\sum_x e^{-\alpha n(x)})^2} \\ &= \langle n^2 \rangle - \langle n \rangle^2 \\ &= (\Delta n)^2.\end{aligned}$$

Thus

$$\begin{aligned}(\Delta n)^2 &= -\frac{\partial \langle n \rangle}{\partial \alpha} \\ &= -\frac{\partial}{\partial \alpha} \left(\frac{6}{e^\alpha - 6} \right) \\ &= \frac{6e^\alpha}{(e^\alpha - 6)^2}\end{aligned}$$

and

$$\frac{\Delta n}{\langle n \rangle} = \sqrt{\frac{e^\alpha}{6}} = \sqrt{\frac{1}{\langle n \rangle} + 1}.$$

Thus the relative dispersion decreases for longer polymers (smaller values of α).