Totally geodesic surfaces in arithmetic hyperbolic 3-manifolds

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Motivation: Length spectra of hyperbolic 3-manifolds

The geometric genus spectrum of a hyperbolic 3-manifold

What is an arithmetic hyperbolic manifold?
Definition: A hyperbolic 3-manifold is a quotient $M = \mathbb{H}^3/\Gamma$ of three dimensional hyperbolic space $\mathbb{H}^3$ by a discrete subgroup $\Gamma$ of $\text{PSL}_2(\mathbb{C})$ acting freely and properly discontinuously.

The Kleinian group $\Gamma$ is isomorphic to the fundamental group $\pi_1(M)$.

If we relax the requirement that $\Gamma$ acts freely, allowing $\Gamma$ to contain torsion, then we obtain a hyperbolic 3-orbifold.

Theorem (Mostow-Prasad Rigidity, 1974)

If $M_1$ and $M_2$ are complete finite volume hyperbolic $n$-manifolds with $n > 2$ then any isomorphism $f : \pi_1(M_1) \to \pi_1(M_2)$ is induced by a unique isometry from $M_1$ to $M_2$. 
Fundamental domains of a pair of isospectral hyperbolic 3-orbifolds
What is an arithmetic hyperbolic 3-manifold?

The **commensurator** $C_\Gamma$ of a Kleinian group $\Gamma \subset \text{PSL}_2(\mathbb{C})$ is

$$C_\Gamma = \{ g \in \text{PSL}_2(\mathbb{C}) : g\Gamma g^{-1} \text{ is commensurable with } \Gamma \}.$$ 

**Theorem (Margulis)**

$\Gamma$ is **arithmetic** if and only if $C_\Gamma$ is dense in $\text{PSL}_2(\mathbb{C})$. 

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Background
Recall the classification of elements $\gamma \in \text{PSL}_2(\mathbb{C}) \setminus \{\text{Id}_2\}$:

- $\gamma$ is elliptic if $\text{Tr}(\gamma) \in \mathbb{R}$ and $|\text{Tr}(\gamma)| < 2$.
- $\gamma$ is parabolic if $\text{Tr}(\gamma) = \pm 2$.
- $\gamma$ is loxodromic otherwise.

We will typically abuse notation and consider the eigenvalues (up to sign) of a lift of $\gamma$ to $\text{SL}_2(\mathbb{C})$. These are the roots of

$$p_\gamma(x) = x^2 - \text{Tr}(\gamma)x + 1;$$

that is,

$$\lambda_\gamma = \frac{\text{Tr}(\gamma) \pm \sqrt{\text{Tr}(\gamma)^2 - 4}}{2}. $$
When $\gamma$ is loxodromic it has a pair of fixed points and the unique geodesic in $H^3$ joining these points is the axis of $\gamma$.

Let $M = H^3/\Gamma$ be a finite-volume hyperbolic 3-manifold. The axis of $\gamma$ projects onto a closed geodesic in $M$ whose length is the translation length $\ell_0(\gamma)$ of $\gamma$, where

$$\ell_0(\gamma) = 2 \log |\lambda_\gamma|.$$ 

The element $\gamma$ also rotates around its axis as it translates along it. If $\theta(\gamma)$ is the angle incurred it translating along the axis by $\ell_0(\gamma)$, then the complex translation length of $\gamma$ is

$$\ell(\gamma) = \ell_0(\gamma) + i\theta(\gamma).$$
The length spectrum $L(M)$ of a hyperbolic 3-manifold $M$ is the set of all complex translation lengths of all closed geodesics in $M$, considered with multiplicities.

The length spectrum of $M$ determines the Laplace spectrum of $M$, hence determines spectral invariants like dimension and volume.

It is known however, that the length spectrum $L(M)$ of $M$ does not determine the isometry class of $M$. 
Theorem (Vignéras, 1980)

There exist non-isometric hyperbolic 3-manifolds with the same length spectra.
It is in general unknown whether $L(M)$ determines the commensurability class of $M$.

This is known to be the case when $M$ is arithmetic however.

Theorem (Chinburg, Hamilton, Long and Reid, 2008)

*If two arithmetic hyperbolic 3-manifolds have the same length spectra then they are commensurable.*
On the other hand non-commensurable hyperbolic 3-manifolds may share arbitrarily large portions of their length spectra.

**Theorem (Futer and Millichap, 2016)**

For every sufficiently large $n > 0$ there exists a pair of non-isometric finite-volume hyperbolic 3-manifolds $\{N_n, N_{n}^{\mu}\}$ such that:

1. \(\text{vol}(N_n) = \text{vol}(N_{n}^{\mu})\), where this volume grows coarsely linearly with $n$.
2. The (complex) length spectra of $N_n$ and $N_{n}^{\mu}$ agree up to length $n$.
3. $N_n$ and $N_{n}^{\mu}$ have at least $e^n/n$ closed geodesics up to length $n$.
4. $N_n$ and $N_{n}^{\mu}$ are not commensurable.

This builds on previous work of Millichap.
The Length Spectrum of Hyperbolic 3-Manifolds

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One of the major open problems in the study of arithmetic hyperbolic 3-manifolds is the following.

**Conjecture (Short Geodesic Conjecture)**

There is a positive universal lower bound for the length of closed geodesics on an arithmetic hyperbolic 3-orbifold.

It is known that the Short Geodesic Conjecture would follow from Lehmer’s Conjecture on Mahler measures of polynomials.

This conjecture has long been known to be false in the context of non-arithmetic hyperbolic 3-orbifolds. In 2006 Agol showed that closed hyperbolic 4-manifolds may also have arbitrarily short closed geodesics.
Let $M$ be a closed hyperbolic 3-manifold.

The length spectrum of $M$ encodes isometric immersions of $S^1$ into $M$.

It turns out to be useful to consider the two-dimensional case; that is, totally geodesic immersions of orientable, finite type surfaces into $M$.

Let $GS(M)$ denote the set of isometry classes of finite area, properly immersed, totally geodesic surfaces of $M$, considered up to free homotopy.

$GS(M)$ is called the **Geometric Genus Spectrum** of $M$. 
The geometric genus spectrum of a hyperbolic 3-manifold

The geometric genus spectrum was introduced by McReynolds and Reid.

**Theorem (McReynolds and Reid, 2009)**

*If two arithmetic hyperbolic 3-manifolds have the same geometric genus spectra then they are commensurable.*
Recently Jeff Meyer and I have proven that non-commensurable hyperbolic 3-manifolds may share arbitrarily large portions of their geometric genus spectra.

This is a two-dimensional analog of Futer and Millichap’s result.

Given $N \geq 1$, define $GS(M)[N]$ to be the first $N$ terms of $GS(M)$ (which we consider as being ordered by area).
The geometric genus spectrum of a hyperbolic 3-manifold

Theorem (L. and Meyer, 2016)

Let \( N \geq 1 \). There exists an infinite sequence of incommensurable arithmetic \( M_1, M_2, \ldots \) such that:

1. \( \text{GS}(M_i)[N] = \text{GS}(M_j)[N] \) for all \( i, j, \)
2. \( \text{vol}(M_n) < c_1(n \log(2n))^{3/2}, \) and
3. \( \text{sys}_1(M_n) < c_2 \log(n). \)
Define $\text{Sys}_{2}^{TG}(M)$ to be the totally geodesic 2-systole of $M$. That is, the minimal area of an immersed totally geodesic surface.

In analogy with the Short Geodesic Conjecture, one may ask whether there is a universal lower bound for $\text{Sys}_{2}^{TG}(M)$ as $M$ varies over all arithmetic hyperbolic 3-orbifolds.

This turns out to be trivially true, as it has long been known that the $(2, 3, 7)$ triangle group has minimal co-area amongst all Fuchsian groups.
The geometric genus spectrum of a hyperbolic 3-manifold

**Theorem (L. and Meyer, 2016)**

Let $M$ be an arithmetic hyperbolic 3-manifold which has minimal volume in its commensurability class and contains a finite area, properly immersed, totally geodesic surface. Then

$$\text{Sys}^2_{TG}(M) > c \cdot \text{vol}(M)^{1/2},$$

where $c$ is a positive constant.

**Corollary**

For every $X > 0$ there exist infinitely many arithmetic hyperbolic 3-manifolds $M$ such that $\text{Sys}^2_{TG}(M) > X$. 
Let $\text{sysg}(M)$ denote the **systolic genus** of $M$; that is, the minimal genus of a $\pi_1$-injective surface of $M$.

Denote by $N(V)$ the number of commensurability classes of arithmetic hyperbolic 3-manifolds which have a representative with volume less than $V$.

Denote by $N^g(V; x)$ the number of commensurability classes of arithmetic hyperbolic 3-manifolds which have a representative $M$ with volume less than $V$ and $\text{sysg}(M) < x$. 
The geometric genus spectrum of a hyperbolic 3-manifold

Theorem (L. and Meyer)

For all sufficiently large $x$ we have

$$\lim_{V \to \infty} \frac{N^g(V; x)}{N(V)} = 0.$$ 

Proof.

Our proof has two main ingredients. The first is a strengthening of Gromov’s high genus systole inequality and is due to Belolieptsy.

The second is a systole counting result which is joint work with Ben McReynolds, Paul Pollack and Lola Thompson.
Theorem (Belolipetsky)

Let $M$ be a closed hyperbolic 3-manifold. For any $\epsilon > 0$, if $\text{sys}_1(M)$ is sufficiently large, then

$$\text{sys}_g(M) \geq e^{(1/2 - \epsilon) \text{sys}_1(M)}.$$ 

Choose $x_0$ large enough so that Belolipetsky’s bound holds with $\epsilon = 1/4$ whenever $\text{sys}_1(M) > x_0$.

It is now straightforward to show that $N^g(V; x)$ is at most the number of commensurability classes of arithmetic hyperbolic 3-manifolds having a representative with volume less than $V$ and systole at most $\max\{x_0, 4 \log(x)\}$. 
The geometric genus spectrum of a hyperbolic 3-manifold

The result now follows from the following.

**Theorem (L., McReynolds, Pollack and Thompson, 2015)**

Let $F(V, X)$ denote the number of commensurability classes of arithmetic hyperbolic 3-manifolds with volume less than $V$ and systole less than $X$. Then

$$\lim_{V \to \infty} \frac{F(V, X)}{N(V)} = 0.$$
What is an arithmetic hyperbolic manifold?
What is an arithmetic hyperbolic manifold?

Let $d$ be a square-free integer and $\mathcal{O}_d$ be the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

The group $\text{PSL}_2(\mathcal{O}_d)$ is a discrete subgroup of $\text{PSL}_2(\mathbb{C})$.

The quotient space $Q_d = \mathbb{H}^3/\text{PSL}_2(\mathcal{O}_d)$ is a non-compact hyperbolic 3-orbifold with finite volume called a Bianchi orbifold.

Theorem

A non-compact hyperbolic 3-orbifold with finite volume is arithmetic if and only if it is commensurable to a Bianchi orbifold.
What is an arithmetic hyperbolic manifold?

Luigi Bianchi

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What is an arithmetic hyperbolic manifold?

The construction of compact arithmetic hyperbolic 3-orbifolds is more nuanced and involves generalizing the following construction of $\text{PSL}_2(\mathcal{O}_d)$.

\[ M_2(\mathbb{Q}(\sqrt{-d})) \supset M_2(\mathcal{O}_d) \rightarrow \text{SL}_2(\mathcal{O}_d) \rightarrow \text{PSL}_2(\mathcal{O}_d). \]

We will replace $M_2(\mathbb{Q}(\sqrt{-d}))$ with a quaternion algebra, $M_2(\mathcal{O}_d)$ with a quaternion order and the determinant map with the reduced norm.
What is an arithmetic hyperbolic manifold?

A brief introduction to quaternion algebras and orders
In the 1830s and 1840s William Rowan Hamilton sought a number system which would play a role in three-dimensional geometry analogous to that of the complex numbers for two-dimensional geometry.

“Every morning in the early part of the above-cited month [October 1843], on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: Well, Papa, can you multiply triplets? Whereto I was always obliged to reply, with a sad shake of the head: No, I can only add and subtract them.”

– Hamilton (in a letter to his son)
What is an arithmetic hyperbolic manifold?

Theorem (Hamilton, 1843)

The $\mathbb{R}$-algebra $\mathbb{H}$ with basis $\{1, i, j, ij\}$ and defining relations

\[ i^2 = -1 \quad j^2 = -1 \quad ij = -ji \]

is a four-dimensional division algebra.

Hamilton was so excited by this discovery that he carved these relations into the stone of the Brougham Bridge.
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What is an arithmetic hyperbolic manifold?

Let’s write $(-1, -1, \mathbb{R})$ in place of $\mathbb{H}$.

This notation suggests a number of ways to generalize $\mathbb{H}$.

For instance, let $(1, 1, \mathbb{R})$ be the $\mathbb{R}$-algebra with basis \{1, i, j, ij\} and defining relations

$$i^2 = 1 \quad j^2 = 1 \quad ij = -ji.$$

Then $(1, 1, \mathbb{R}) \cong M_2(\mathbb{R})$ via $i \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $j \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
More generally, we can define \((a, b, \mathbb{R})\) to be the \(\mathbb{R}\)-algebra with basis \(\{1, i, j, ij\}\) and defining relations

\[
i^2 = a \quad j^2 = b \quad ij = -ji \quad a, b \in \mathbb{R}^*.
\]

It’s not too hard to show that

- \((a, b, \mathbb{R}) \cong \mathbb{H}\) if \(a, b < 0\) and
- \((a, b, \mathbb{R}) \cong \mathbb{M}_2(\mathbb{R})\) otherwise.

Thus \((a, b, \mathbb{R})\) is either a division algebra or else \(\mathbb{M}_2(\mathbb{R})\).
What is an arithmetic hyperbolic manifold?

There are other ways that we could have generalized \((-1, -1, \mathbb{R})\).

If \(F\) is a field of characteristic zero and \(a, b \in F^*\) we can define the generalized quaternion algebra \((a, b, F)\).

Let \(F = \mathbb{Q}\) and consider the \(\mathbb{Q}\)-algebra \((-1, -1, \mathbb{Q})\).

As \((-1, -1, \mathbb{Q}) \subset (-1, -1, \mathbb{R})\), we see that \((-1, -1, \mathbb{Q})\) is a division algebra.

As before we also see that \((1, 1, \mathbb{Q}) \cong M_2(\mathbb{Q})\).
What is an arithmetic hyperbolic manifold?

Recall that if $F = \mathbb{R}$, $(a, b, F)$ is a division algebra or else $M_2(\mathbb{R})$.

**Theorem (Wedderburn)**

For any field $F$, if the $F$-algebra $(a, b, F)$ is not a division algebra then $(a, b, F) \cong M_2(F)$.

Note that the case $F = \mathbb{R}$ is special in that in general there will not be a unique quaternion division algebra over $F$. 
What is an arithmetic hyperbolic manifold?

A “complex” example: \( F = \mathbb{C} \).

Let \( A \) be a quaternion algebra over \( \mathbb{C} \).

By the fundamental theorem of algebra, \( A \) cannot be a division algebra.

Then by Wedderburn’s theorem, \( A \cong M_2(\mathbb{C}) \).

Thus \( M_2(\mathbb{C}) \) is the only quaternion algebra over \( \mathbb{C} \).
Extension of scalars:

Let $F$ be a field and $F'/F$ a field extension.

If $A$ is a quaternion algebra over $F$ then we can consider the quaternion algebra $A \otimes_F F'$ over $F'$.

Concretely, if $A = (a, b, F)$ then $A \otimes_F F' = (a, b, F')$.

This is especially important in arithmetic applications.
Reduced norm:

Let $A$ be a quaternion algebra over a number field $F$.

We’ve already seen that we have an embedding $A \hookrightarrow M_2(\mathbb{C})$.

The **reduced norm** of $A$ is the composite map

$$A \hookrightarrow M_2(\mathbb{C}) \to^{\text{det}} \mathbb{C}$$

For $A = M_2(F)$ the reduced norm coincides with the determinant.
What is an arithmetic hyperbolic manifold?

Let $F$ be a number field with ring of integers $\mathcal{O}_F$.

An order of an $F$-algebra is a subring which is also a finitely generated $\mathcal{O}_F$-module containing an $F$-basis of the algebra.

**Example 1:** $\mathbb{Z}[i]$ is a quadratic order of the $\mathbb{Q}$-algebra $\mathbb{Q}(i)$.

**Example 2:** $\text{M}_2(\mathbb{Z})$ is a maximal order of $\text{M}_2(\mathbb{Q})$.

**Example 3:** $\mathcal{O}_F[i, j]$ is always an order of the $F$-algebra $(a, b, F)$ when $a, b \in \mathcal{O}_F$. 
What is an arithmetic hyperbolic manifold?

Constructing hyperbolic 3-orbifolds from quaternion orders
What is an arithmetic hyperbolic manifold?

Consider the imaginary quadratic field \( \mathbb{Q}(\sqrt{-3}) \) and quaternion algebra \( B = (-1, 7, \mathbb{Q}(\sqrt{-3})) \).

There is an isomorphism \( \varphi : B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{C}) \).

Let \( \mathcal{O} \) be a maximal orders of \( B \) and \( \mathcal{O}^1 \) be the multiplicative subgroup consisting of those elements of reduced norm 1.

The image in \( \text{PSL}_2(\mathbb{C}) \) of \( \mathcal{O}^1 \) is denoted \( \Gamma^1_{\mathcal{O}} \) and is a cocompact arithmetic Kleinian group.

One can define an \textbf{arithmetic hyperbolic 3-manifold} to be a hyperbolic 3-manifold whose fundamental group is isomorphic to a group of the form \( \Gamma^1_{\mathcal{O}} \) (though perhaps with a different number field and quaternion algebra).
What is an arithmetic hyperbolic manifold?

**Theorem (Borel)**

The volume of $H^3/\Gamma^1_\mathcal{O}$ is

$$\text{vol}(H^3/\Gamma^1_\mathcal{O}) = \frac{|d_{-3}|^{3/2} \zeta_{-3}(2)}{4\pi^2} \prod_{p \in \text{Ram}_f(B)} (N(p) - 1)$$

$$= \frac{|-3|^{3/2} \cdot 1.28519}{4\pi^2} (7 - 1)(7 - 1)$$

$$= 6.08964\ldots$$
What is an arithmetic hyperbolic manifold?
What is an arithmetic hyperbolic manifold?

Now consider the rational quaternion algebra $A = (-1, 7, \mathbb{Q})$.

There is an isomorphism $\varphi : A \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$.

Let $\mathcal{E}$ be a maximal orders of $A$ and $\mathcal{E}^1$ be the multiplicative subgroup consisting of those elements of reduced norm 1.

The image in $\text{PSL}_2(\mathbb{R})$ of $\mathcal{E}^1$ is denoted $\Gamma_{\mathcal{E}}^1$ and is a cocompact arithmetic Fuchsian group.

Because

$$A \otimes_{\mathbb{Q}} \mathbb{Q}(-3) = (-1, 7, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(-3) = (-1, 7, \mathbb{Q}(-3)) = B,$$

we get an inclusion $\Gamma_{\mathcal{E}}^1 \subset \Gamma_{\mathcal{O}}^1$. 
The area of $H^2/\Gamma^1_\mathcal{E}$ is

$$\text{area}(H^2/\Gamma^1_\mathcal{E}) = \frac{8\pi |d_\mathbb{Q}|^{3/2} \zeta(2)}{4\pi^2} \prod_{p \in \text{Ram}_f(A)} (N(p) - 1)$$

$$= \frac{8\pi \cdot 1^{3/2} \cdot \pi^2 / 6}{4\pi^2} (2 - 1)(7 - 1)$$

$$= 2\pi.$$
In this manner we obtain a hyperbolic 3-orbifold $\mathbb{H}^3/\Gamma^1_O$ with a totally geodesic surface $\mathbb{H}^2/\Gamma^1_\mathcal{E}$ of area $2\pi$.

The commensurability class of an arithmetic hyperbolic surface / 3-manifold is given by the quaternion algebra associated to it.

Therefore any other surface arising from the quaternion algebra $A = (-1, 7, \mathbb{Q})$ would be commensurable to $\mathbb{H}^2/\Gamma^1_\mathcal{E}$. 
What is an arithmetic hyperbolic manifold?

So in order to produce incommensurable totally geodesic surfaces we want non-isomorphic quaternion algebras $A_1, A_2$ such that

$$A_1 \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-3}) \cong (-1, 7, \mathbb{Q}(\sqrt{-3})) \cong A_2 \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-3}).$$

Similarly, to produce non-commensurable hyperbolic 3-manifolds with a totally geodesic surface in common, we want to choose quaternion algebras like $(-1, 7, \mathbb{Q}(\sqrt{-d_1}))$ and $(-1, 7, \mathbb{Q}(\sqrt{-d_2}))$ (for $d_1 \neq d_2$). Kleinian groups arising from both of these algebras will contain Fuchsian groups like $\Gamma^1_E$. 

What is an arithmetic hyperbolic manifold?

Producing non-commensurable arithmetic hyperbolic 3-manifolds with large overlaps in their geometric genus spectra is more nuanced.

It amounts to finding a large number of non-isomorphic quaternion algebras $A_1, A_2, \ldots$ over $\mathbb{Q}$ and two quaternion algebras $B_1, B_2$ over $\mathbb{Q}(\sqrt{-d_1}), \mathbb{Q}(\sqrt{-d_2})$ such that

$$A_1 \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-d_1}) \cong A_2 \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-d_1}) \cong \cdots \cong B_1$$

and

$$A_1 \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-d_2}) \cong A_2 \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-d_2}) \cong \cdots \cong B_2.$$ 

Our quantitative estimates require relating the volume formula for surfaces arising from the $A_i$ to 3-manifolds arising from $B_1$ and $B_2$. 
Thanks!