Gaussian Integers and Arctangent Identities for π

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1. INTRODUCTION. The Gregory series for arctangent

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \quad |x| \leq 1,$$

combines with the identity

$$\frac{\pi}{4} = \arctan 1$$

to yield Leibniz's series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

This series was a theoretical breakthrough in the calculation of decimal digits of π, although it is impractical due to its excruciatingly slow rate of convergence. Elegant identities such as

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}, \qquad \text{(Machin, 1706 [**48**, p. 7])}$$

$$\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}, \qquad \text{(Euler, 1755 [**55**, p. 645])}$$

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}, \qquad \text{(Machin, 1706 [**20, 48**])}$$

$$\frac{\pi}{2} = 2\arctan\frac{1}{2} + \arctan\frac{4}{7} + \arctan\frac{1}{8}, \text{ and} \qquad \text{(Newton, 1676 [**48**, p. 2])}$$

$$\frac{\pi}{4} = 8\arctan\frac{1}{10} - \arctan\frac{1}{239} - 4\arctan\frac{1}{515} \qquad \text{(Simson, 1723 [**48**, p. 10])}$$

utilize the x^n terms in Gregory's series and have been instrumental in the calculation of decimal digits of π. Upon learning of these identities, one naturally desires an identity of the form

$$\pi = r \arctan x$$

where r and x are rational and $|x| < 1$ is small. Such an identity would require only one evaluation of the arctangent function and this evaluation would converge quickly. However, identities of this form do not exist and this fact is not mentioned in the literature alongside lists of such multiple-angle identities. The present article gives a very natural proof of this fact using a simple consequence of unique factorization of Gaussian integers (Main Lemma, Section 2). Section 3 gives several applications of the Main Lemma to arctangent identities, triangles, polygons on geoboards, and smooth curves without rational points. Section 4 concludes with historical remarks and suggestions for further reading. Two key points for the reader to take away are: the complex numbers can provide insight into real ($\mathbb{R}$) problems, and unique factorization, when it exists, is a powerful tool. Indeed, a fallacious proof of Fermat's last theorem given by Lamé in 1847 depended crucially on unique factorization in certain rings where unique factorization fails. This failure led to important advances in algebra (see [**38**, pp. 4–5, Ch. 5] or [**41**, pp. 169–176]).

2. GAUSSIAN INTEGERS. To make this article self contained, we review basic facts about the Gaussian integers. The reader familiar with the Gaussian integers should skip ahead to the Main Lemma at the end of this section. In this paper the natural numbers $\mathbb{N}$ consist of the positive integers.

Let R be a commutative ring with additive and multiplicative identities $0 = 0_R$ and $1 = 1_R$ respectively. Say that $w \neq 0$ *divides* z in R, written $w|z$, provided there exists $v \in R$ such that $wv = z$. A nonzero element $w \in R$ is called a *zero divisor* if there exists a nonzero element $v \in R$ such that $wv = 0$ (for example, $[2]\,[3] = [0]$ in $\mathbb{Z}/6\mathbb{Z}$). R is an *integral domain* provided it contains no zero divisors, and this is equivalent to cancellation holding in R: $wv = wz$ and $w \neq 0$ imply $v = z$.

A *unit* in R is a divisor of 1. Elements w and z in R are *associates* provided $w|z$ and $z|w$. Cancellation implies that (nonzero) elements are associates if and only if they differ by multiplication by a unit. An element $z \in R$ is *irreducible* provided it is a nonzero nonunit and if $z = wv$, then w or v is a unit; in other words, z admits no nontrivial factorization. An element $z \in R$ is *prime* provided it is a nonzero nonunit and if $z|wv$, then $z|w$ or $z|v$.

An integral domain R is a *unique factorization domain* (UFD) provided every nonzero nonunit in R is the product of finitely many irreducibles in R (existence) and this product is unique up to order and unit multiples (uniqueness).

The rational integers $\mathbb{Z}$ form the canonical and motivating example of a UFD. We remark that the terminology "rational integer" is standard; one may define the ring of integers in any number field K and if $K = \mathbb{Q}$ is the rational field, then the ring of integers is $\mathbb{Z}$ [**38**, §2.3]. Commonly, one calls a rational integer p "prime" provided it is not equal to 0 or ± 1 and its only rational integer divisors are ± 1 and $\pm p$. This abuse of terminology (such a p is technically irreducible) is overlooked since irreducibles and primes coincide in $\mathbb{Z}$; a proof of this fact, and that $\mathbb{Z}$ is a UFD, follows exactly the same steps as below for the Gaussian integers. Note that in any integral domain, primes are irreducibles by cancellation; however irreducibles need not be primes. For example, working in $\mathbb{Z}\left[\sqrt{-3}\right] = \left\{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\right\}$ we have

$$2 \cdot 2 = \left(1 + \sqrt{-3}\right)\left(1 - \sqrt{-3}\right). \tag{1}$$

The norm

$$N\left(a + b\sqrt{-3}\right) = \left(a + b\sqrt{-3}\right)\left(a - b\sqrt{-3}\right) = a^2 + 3b^2$$

is multiplicative, meaning $N(\alpha\beta) = N(\alpha)N(\beta)$, from which it follows easily that $1 + \sqrt{-3}$ is irreducible. Equation (1) shows that $1 + \sqrt{-3}$ divides the product $2 \cdot 2$. However, it is easy to check directly that $1 + \sqrt{-3}$ does not divide 2. Therefore, $1 + \sqrt{-3}$ is irreducible but not prime in $\mathbb{Z}\left[\sqrt{-3}\right]$. For more examples see [**38**, Section 4.4]. We mention that non-prime irreducibles are completely characteristic of integral domains in which factorization into irreducibles exists but is not unique [**38**, Theorem 4.13].

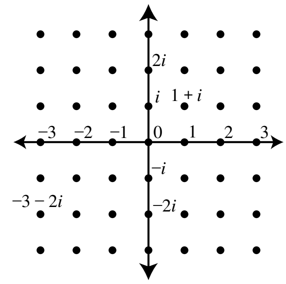

Figure 1. The Gaussian integers $\mathbb{Z}[i]$, a square lattice in $\mathbb{C}$.

The *Gaussian integers* $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ are the integer lattice in $\mathbb{C}$ (see Figure 1) and form a commutative ring. The *norm* of $z = a + bi$ is $N(z) = z\overline{z} = a^2 + b^2$. The norm is multiplicative: $N(zw) = N(z)N(w)$. In particular, if $w|z$ in $\mathbb{Z}[i]$, then $N(w)|N(z)$ in $\mathbb{Z}$.

The norm shows that $\mathbb{Z}[i]$ contains no zero divisors. Hence it is an integral domain and cancellation holds, and the units in $\mathbb{Z}[i]$ are ± 1 and $\pm i$. The norm will also play a key role in showing $\mathbb{Z}[i]$ is a UFD. One should note that the square shape of the lattice $\mathbb{Z}[i]$ is at the heart of unique factorization. For interesting discussions on lattice shape and the success or failure of unique factorization in certain integral domains, see [**40**, pp. 229–245] and [**41**, pp. 169–176].

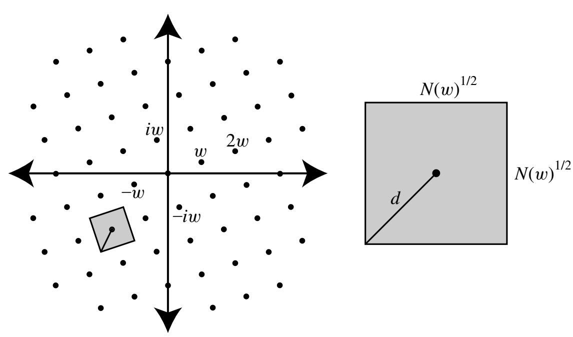

Figure 2. Square geometry of a sublattice yielding the division property.

The first step to unique factorization in $\mathbb{Z}[i]$ is to prove the *division property*: if $w \neq 0$ and z are Gaussian integers, then there exist Gaussian integers φ and ρ such that $z = \varphi w + \rho$ and $N(\rho) < N(w)$. For the proof, observe that the sublattice

$$w\mathbb{Z}[i] = \{\varphi w \mid \varphi \in \mathbb{Z}[i]\} \subseteq \mathbb{Z}[i]$$

has a square shape as shown in Figure 2. Therefore each $z \in \mathbb{Z}[i]$ lies within d units of a point in $w\mathbb{Z}[i]$ where $d = \sqrt{N(w)}/\sqrt{2} < \sqrt{N(w)}$. Choose φ such that $N(z - \varphi w)$ is minimized (such a φ may not be unique, but that does not matter) and let $\rho = z - \varphi w$.

The next step is to obtain the greatest common divisor (gcd) via the Euclidean algorithm, which we now review in the context of Gaussian integers. If w and z are Gaussian integers, not both zero, then we define $\gcd(w, z)$ to be any common divisor of w and z of maximal norm. As w or z is nonzero, the set of common divisors is finite (since the norm is multiplicative and nonnegative) and contains elements of maximal norm, so $\gcd(w, z)$ exists. By taking unit multiples of one value of $\gcd(w, z)$ we obtain four values of $\gcd(w, z)$. It seems possible that $\gcd(w, z)$ may assume more than four values, that is, two values of $\gcd(w, z)$ might not be associates. We will see this is not the case. With $\gcd(w, z)$ defined as above, we will prove that:

(i) $\gcd(w, z)$ is unique up to multiplication by units,

(ii) every common divisor of w and z divides $\gcd(w, z)$, and

(iii) $\gcd(w, z)$ is a Gaussian integer linear combination of w and z.

Property (i) says that $\gcd(w, z)$ assumes exactly four values. We will prove these properties by producing $\gcd(w, z)$ via the Euclidean algorithm.

The core idea in the Euclidean algorithm is the following.

Euclidean Algorithm Core Idea. Let α and β be nonzero Gaussian integers. By the division algorithm, write $\beta = \varphi\alpha + \rho$ with $N(\rho) < N(\alpha)$. Then

(a) the pair $\{\beta, \alpha\}$ has the same set of common divisors as the pair $\{\alpha, \rho\}$, and

(b) the pair $\{\beta, \alpha\}$ has the same set of Gaussian integer linear combinations as the pair $\{\alpha, \rho\}$.

The proofs of (a) and (b) are very simple. For the backward direction in (b), let $\gamma = x\alpha + y\rho$ be a Gaussian integer linear combination of α and ρ. Then $\gamma = y\beta + (x - y\varphi)\alpha$ is a Gaussian integer linear combination of β and α. The other three parts are proved similarly.

The Euclidean algorithm starts with Gaussian integers w and z, not both zero, and produces $\gcd(w, z)$. In case, say, $w = 0$, then the output is z. Otherwise, w and z are nonzero and without loss of generality $N(w) \leq N(z)$. Apply the division algorithm repeatedly as follows:

$$\begin{aligned}
z &= \varphi_1 w + \rho_1 && \text{such that } 0 < N(\rho_1) < N(w),\\
w &= \varphi_2 \rho_1 + \rho_2 && \text{such that } 0 < N(\rho_2) < N(\rho_1),\\
\rho_1 &= \varphi_3 \rho_2 + \rho_3 && \text{such that } 0 < N(\rho_3) < N(\rho_2),\\
&\cdots && \cdots \qquad\qquad \cdots\\
\rho_{k-2} &= \varphi_k \rho_{k-1} + \rho_k && \text{such that } 0 < N(\rho_k) < N(\rho_{k-1}), \text{ and}\\
\rho_{k-1} &= \varphi_{k+1} \rho_k + 0.
\end{aligned}$$

The procedure halts when a remainder of 0 is first obtained (last line above) and the output is the last nonzero remainder ρ_k. The procedure halts after finitely many steps since the norms of successive remainders form a strictly decreasing sequence of nonnegative integers. We now show that the output is $\gcd(w, z)$ and that the gcd satisfies the three properties stated above. Starting with the pair $\{z, w\}$, apply part (a) of the Euclidean Algorithm Core Idea $k + 1$ times to see that the pairs

$$\{z, w\},\ \{w, \rho_1\},\ \{\rho_1, \rho_2\},\ \ldots,\ \{\rho_{k-1}, \rho_k\},\ \text{and } \{\rho_k, 0\}$$

all have the same set of common divisors. In other words, the set of common divisors of z and w is precisely the set of divisors of ρ_k. The norm is multiplicative and nonnegative, and so the divisors of ρ_k of maximal norm are exactly the associates of ρ_k. Thus, the values of $\gcd(w, z)$ are exactly the associates of ρ_k, and property (i) is proved. Property (ii) is now immediate. To prove property (iii), we simply work backwards. Clearly ρ_k is a linear combination of the pair $\{\rho_k, 0\}$. After $k+1$ applications of part (b) of the Euclidean Algorithm Core Idea, we obtain ρ_k as a Gaussian integer linear combination of the pair $\{z, w\}$ as desired. Note that in the trivial case $w = 0$, the values of $\gcd(0, z)$ are exactly the associates of z and the three properties are satisfied. Thus $\gcd(w, z)$ has properties (i)–(iii).

If w and z are Gaussian integers and $\gcd(w, z)$ is a unit, then w and z are said to be *relatively prime*. In this case, properties (i) and (iii) imply that there exist Gaussian integers x and y such that $xw + yz = 1$.

With the gcd in hand, one may show that irreducibles are primes in $\mathbb{Z}[i]$. Let $w \in \mathbb{Z}[i]$ be irreducible. Then the set of divisors of w equals $\{\pm 1, \pm i, \pm w, \pm iw\}$ and we have the following.

Observation. If α is a Gaussian integer, then $\gcd(w, \alpha) = 1$ or $\gcd(w, \alpha) = w$ according to whether α divides w.

So, suppose $w|zv$ in $\mathbb{Z}[i]$. By the observation, either $\gcd(w, z) = 1$ or $\gcd(w, z) = w$. The latter implies $w|z$, while the former implies $xw + yz = 1$ for some Gaussian integers x and y; thus, $xwv + yzv = v$ and so $w|v$ since $w|zv$, as desired. Therefore, irreducibles and primes coincide in $\mathbb{Z}[i]$ (recall that primes are irreducibles in any integral domain by cancellation).

It follows easily that $\mathbb{Z}[i]$ is a UFD: each nonzero nonunit w has a factorization into irreducibles by induction on $N(w)$, and this factorization is unique up to order and unit multiples using cancellation repeatedly and the fact that irreducibles are primes.

A thorough introduction to the Gaussian integers would include the characterization of Gaussian primes. We will not need this characterization, so the interested reader may see [**40**, pp. 233–236] or most any elementary number theory text. We do, however, point out some basic facts that are used below. If $N(w)$ is a rational prime, then w is a Gaussian prime. In particular, $1 + i$ is prime. The following are equivalent for a Gaussian integer w: w is prime, $\overline{w}$ is prime, and uw is prime where u is any unit.

We now come to the main result of this section. The Gaussian integers that lie on the four lines in Figure 3 will play a key role.

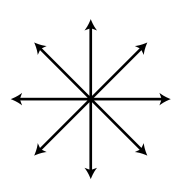

Figure 3. Four lines in $\mathbb{C}$: $\mathrm{Im} z = 0$, $\mathrm{Re} z = \mathrm{Im} z$, $\mathrm{Re} z = 0$, and $\mathrm{Re} z = -\mathrm{Im} z$.

Main Lemma. *Let $z \neq 0$ be a Gaussian integer. There is a natural number n such that z^n is real if and only if z lies on one of the four lines in Figure 3.*

Proof. For the backward direction, let $n = 1$, 2, or 4. For the forward direction, let $z^n = m \in \mathbb{Z}$ where $0 \neq z = a + bi$. The general case follows easily from the case where z is a nonunit and is *primitive*, that is, $\gcd(a, b) = 1$. In this case, let w be any Gaussian prime divisor of z. Then $w|m$ and so $\overline{w}|\overline{m} = m$ since m is real. As $\overline{w}$ is a Gaussian prime that divides $m = z^n$, we see that $\overline{w}|z$. Unique factorization immediately implies the following.

Fact. *If w is a Gaussian prime dividing z, and w and $\overline{w}$ are not associates, then $w\overline{w} \in \mathbb{Z}$ divides z.*

As z is primitive, the fact implies that z is a product of Gaussian primes, each of which is an associate of its conjugate. Let $v = c + di$ be such a Gaussian prime factor of z. As v and $\overline{v}$ are associates, we see that $c = 0$, $d = 0$, or $c = \pm d$. The first two cases are not possible, since z is primitive. The third case implies $c = \pm 1$ since v is prime. It follows that v is an associate of $1 + i$ and $z = u\,(1+i)^l$ for some unit u and natural number l. ■

3. APPLICATIONS: ARCTANGENT IDENTITIES, TRIANGLES, GEOBOARDS, AND SMOOTH CURVES.

The Main Lemma in the previous section has several applications which are presented below. Throughout this section $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Corollary 1. *The only rational values of* $\tan k\pi/n$ *are* 0 *and* ± 1.

Proof. Suppose $\tan k\pi/n = b/a$ where $b \in \mathbb{Z}$ and $a \in \mathbb{N}$. Then

$$\frac{k\pi}{n} = \arg(a+ib) \;\Rightarrow\; k\pi = \arg(a+ib)^n \;\Rightarrow\; (a+ib)^n \in \mathbb{Z}$$

and so every argument of $a + ib$, namely $k\pi/n$, is an integer multiple of $\pi/4$ by the Main Lemma. The result follows. ■

We now come to the nonexistence result on single-angle arctangent identities for π stated in the introduction.

Corollary 2. *Identities of the form* $k\pi = n \arctan x$ *with x rational have $x = 0$ or $x = \pm 1$. In particular,* $\pi = 4 \arctan 1$ *is the most efficient such identity for computing π using Gregory's series.*

Proof. Given $k\pi/n = \arctan x$, apply tan and use the previous corollary. ■

Multiple-angle rational arctangent identities for π have the form

$$\frac{k\pi}{n} = \sum_{j=1}^{l} m_j \arctan \frac{b_j}{a_j} \tag{2}$$

where all variables are rational integers. It is natural to assume, without loss of generality, that $n \in \mathbb{N}$, $\gcd(k, n) = 1$, $\gcd(m_1, \ldots, m_l) = 1$, the values $\left|b_j/a_j\right|$ are distinct, and that for all j: $m_j \in \mathbb{N}$, $b_j \neq 0$, $a_j \in \mathbb{N}$, and $\gcd\left(a_j, b_j\right) = 1$. Note that we allow $k = 0$. Even though nontrivial identities exist with $l \geq 2$, it turns out that one does not obtain any new angles $k\pi/n$ over the $l = 1$ case.

Corollary 3. *If* (2) *holds, then $k\pi/n = j\pi/4$ for some integer j. In particular, $n = 1$, 2, or 4.*

Proof. Modulo 2π we have

$$\frac{k\pi}{n} = \sum_{j=1}^{l} m_j \arctan \frac{b_j}{a_j} = \sum_{j=1}^{l} m_j \arg\left(a_j + ib_j\right) = \arg \prod_{j=1}^{l} \left(a_j + ib_j\right)^{m_j}.$$

Let z denote the product above. Then z^n is real and so $\arg z = k\pi/n$ is a multiple of $\pi/4$ by the Main Lemma. The result follows. ■

The reason nontrivial multiple-angle identities exist is quite simple. Let us look at double-angle identities

$$\frac{k\pi}{n} = m_1 \arctan\frac{b_1}{a_1} + m_2 \arctan\frac{b_2}{a_2}. \tag{3}$$

Let $z_j = a_j + ib_j$ for $j = 1, 2$ and $z = z_1^{m_1} z_2^{m_2}$. Equation (3) implies that z^n is real (conversely, if $\left(z_1^{m_1} z_2^{m_2}\right)^n$ is real, where $z_j = a_j + ib_j$ and $a_j \neq 0$ for $j = 1, 2$, then one obtains an identity of the form (3) for some $k \in \mathbb{Z}$). As z^n is real, the proof of the Main Lemma shows that if $w \in \mathbb{Z}[i]$ is a prime divisor of z, then $\overline{w}$ is as well. The difference now, over the single-angle case, is that these conjugate primes may appear separately in z_1 and z_2. An example is enlightening. Pick a prime, say $2 + i$. Letting $z_1 = 2 + i$, $z_2 = 2 - i$, and $n = m_1 = m_2 = 1$, we have $z^n = 5$. However, the corresponding arctangent identity is useless: $0\pi = \arctan(1/2) + \arctan(-1/2)$. So, introduce a factor of $1 + i$ and let

$$z_2 = (2 - i)(1 + i) = 3 + i$$

and, correspondingly, $n = 4$. Now $z^n = -2500$ and the associated identity is

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}.$$

It is instructive to take the arctangent identities from the introduction and produce their associated Gaussian integer equations (factored into primes). The importance of units and $1 + i$ should become apparent. Several questions arise, such as: are there useful identities without factors of $1 + i$ in z_1 or z_2? A little tinkering yields

$$-78125 = \left[(2 + i)^7 (-1)\right]^1 [2 - i]^7$$

where

$$(2 + i)^7 = -278 - 29i$$

and correspondingly

$$-\pi = \arctan\frac{29}{278} + 7 \arctan\frac{-1}{2}.$$

Thus useful identities without factors of $1 + i$ exist. At this point, modern students should be well equipped to explore multiple-angle identities on their own using a computer. The author highly advocates this personal form of discovery, since it is beneficial and fun! The reader might enjoy finding an efficient identity and checking the literature to see if it is known. The bibliography lists much of the literature known to the author.

The Main Lemma also has several applications to triangles. Say that an angle is *rational* provided it is commensurable with a straight angle; equivalently, its degree measure is rational or its radian measure is a rational multiple of π. Say that a side of a triangle is *rational* provided its length is rational.

Corollary 4. *A right triangle with rational acute angles and rational legs is a 45-45-90 triangle.*

Proof. Suppose triangle $\triangle ABC$ has a right angle at C, rational legs a and b opposite the angles at A and B respectively, and the angle β at B is a rational multiple of π. Then $\tan\beta = b/a$ is rational and equals $+1$ by Corollary 1 since side lengths are positive. Therefore, $\beta = \pi/4$ and the result follows. ■

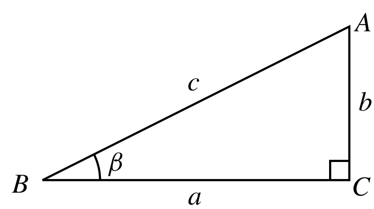

Figure 4. Right triangle $\triangle ABC$.

Corollary 5. *The acute angles in a right triangle with rational side lengths are never rational.*

Proof. Suppose triangle $\triangle ABC$ has a right angle at C, rational side lengths a, b, and c opposite the angles at A, B, and C respectively, and the angle β at B is a rational multiple of π. The previous corollary implies $a = b$. The Pythagorean theorem implies $2a^2 = c^2$, a familiar contradiction to unique factorization of rational integers. ■

In other words, every triangle whose side lengths form a Pythagorean triple has acute angles of irrational degree measure, thus explaining why the angles of such triangles are never emphasized in grade school. Stillwell used our approach to obtain this result in [**41**, pp. 168–169].

Recall that a *geoboard* is a flat board containing pegs in a square lattice shape (see Figure 5). Rubber bands are placed around collections of pegs to form polygons, angles, and so forth. Common questions include whether one can build an equilateral triangle, regular polygons in general, and certain angles on a geoboard. One has to be a little careful here as shown on the right in Figure 5. Each straight segment S of rubber band stretches between two pegs P_1 and P_2; if S is not parallel to the segment connecting the centers of P_1 and P_2, then the angles and polygons one can build depend

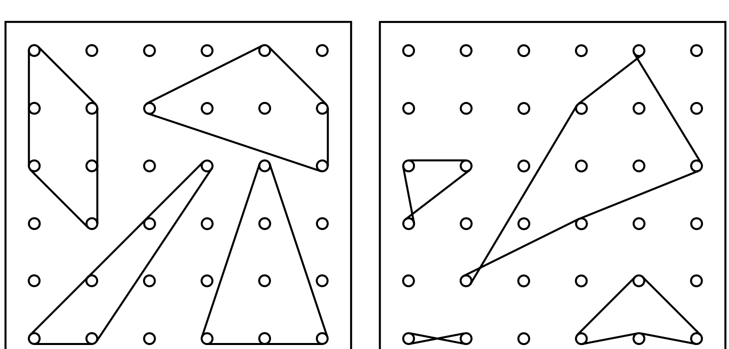

Figure 5. Two 6×6 geoboards: admissible bands (left) and inadmissible bands (right).

on the diameter of the pegs. For precision, one can make this parallel assumption, or idealize and turn the pegs into points. We adopt the latter approach. Define a *lattice angle* to be an angle formed by rays $\overrightarrow{vw}$ and $\overrightarrow{vz}$ where $v, w, z \in \mathbb{Z}[i]$ and $v \neq w, z$ (see Figure 6).

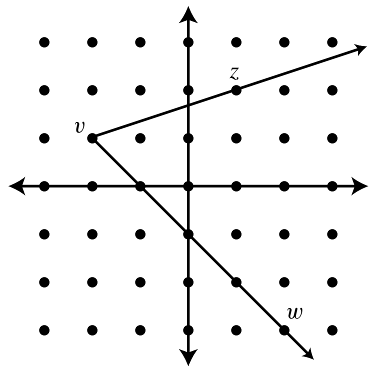

Figure 6. A lattice angle.

Corollary 6. *If a lattice angle is rational, then its measure is an integer multiple of* $\pi/4$.

Proof. Let the rays $\overrightarrow{vw}$ and $\overrightarrow{vz}$ form a rational lattice angle. Our tactic is to apply two angle preserving algebraic transformations of $\mathbb{Z}[i]$ so that the Main Lemma applies to the resulting congruent angle. First, translate by $-v$, which maps $\mathbb{Z}[i]$ into itself and is an isometry of $\mathbb{C}$; let $w' = w - v$ and $z' = z - v$. Second, multiply by $\overline{w'} \neq 0$, which maps $\mathbb{Z}[i]$ into itself and is a similarity transformation. More specifically, this second transformation rotates about 0 by $\arg \overline{w'}$ and scales all lengths by $N\left(\overline{w'}\right)^{1/2}$; let $W = w'\overline{w'}$ and $Z = z'\overline{w'}$. The angle formed by $\overrightarrow{0W}$ and $\overrightarrow{0Z}$ is rational, being congruent to the original angle, and $W > 0$. Therefore, Z^n is real for some natural number n. The result follows by the Main Lemma. ∎

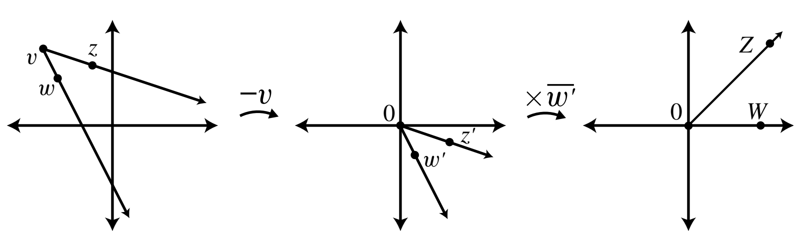

Figure 7. Translation by $-v$ and multiplication by $\overline{w'}$.

For precision, let us take a moment to define relevant terms concerning polygons. A *polygon* consists of $n \geq 3$ vertices $p_1, p_2, \ldots, p_n$ in the plane $\mathbb{C}$, along with the segments $\overline{p_1 p_2}, \overline{p_2 p_3}, \ldots, \overline{p_{n-1} p_n}, \overline{p_n p_1}$ called *edges*; we further assume two natural nondegeneracy conditions: consecutive vertices are distinct (no edge is a point) and consecutive triples of vertices are not collinear. A *lattice polygon* has vertices that are Gaussian integers. A polygon is *simple* provided only consecutive edges intersect and only at their single common vertex. A polygon is *equilateral* provided each of its edges has the same length. A pair of consecutive edges intersecting at p_j defines a *vertex* *angle* whose measure α_j we take to lie in $(0, \pi)$. A polygon is *equiangular* provided each vertex angle has the same measure and, moreover, these angles all turn the same way, either all clockwise or all counterclockwise, as one traverses the polygon along consecutive edges. A polygon is *regular* provided it is equilateral and equiangular. A *regular star* is a regular polygon that is not simple, as in Figure 8.

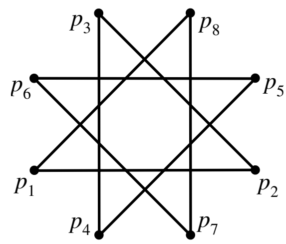

Figure 8. A regular star.

Claim. A regular polygon, not necessarily simple or lattice, has rational vertex angles.

Proof. Let P be a regular polygon in $\mathbb{C}$ with n vertices $p_1, p_2, \dots, p_n$. Translate and rotate so that $p_1 = 0$ and $p_2 = r > 0$. Let $0 < \alpha < \pi$ denote the measure of each vertex angle and $\theta = \pi - \alpha$. Then

$$\begin{aligned} p_2 - p_1 &= r, \\ p_3 - p_2 &= re^{i\theta}, \\ p_4 - p_3 &= re^{i2\theta}, \\ &\vdots \\ p_n - p_{n-1} &= re^{i(n-2)\theta}, \text{ and} \\ p_1 - p_n &= re^{i(n-1)\theta}. \end{aligned}$$

Add these n equations and let $x = \exp(i\theta)$ to obtain

$$0 = r\left(1 + x + x^2 + x^3 + \cdots + x^{n-1}\right).$$

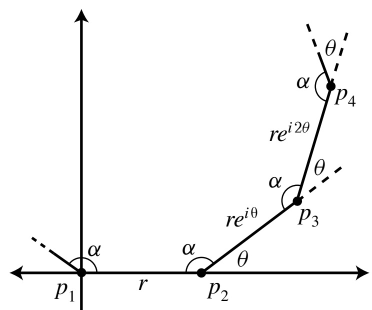

Figure 9. Regular polygon P.

Multiplying by $(x-1)/r$ yields $0 = x^n - 1$. Therefore, $x = \exp(i\theta)$ is an nth root of unity, $\theta = 2\pi k/n$ for some k satisfying $0 < k < n/2$ (since $0 < \theta < \pi$), and $\alpha = \pi(n-2k)/n$ is a rational multiple of π, as desired. ■

Corollary 7. *The only regular lattice polygon is a square. In particular, there does not exist a regular lattice star.*

Proof. Let P be a regular lattice polygon with vertex angles of measure α, $0 < \alpha < \pi$. By the previous claim, α is a rational multiple of π. The previous corollary implies that $\alpha = \pi/4, \pi/2$, or $3\pi/4$. Clearly $\alpha = \pi/2$ yields a square, which exists (in many ways) as a regular lattice polygon. It remains to rule out the other two possibilities. Let p_1 and p_2 be adjacent vertices in P. Translation by the Gaussian integer $-p_1$ allows us to assume $p_1 = 0$. Multiply by the Gaussian integer $\overline{p_2}$ so that an adjacent vertex is on the positive real axis (all lengths scale by $N(\overline{p_2})^{1/2}$). After a possible reflection across the real axis, the resulting lattice polygon P', which is similar to P, appears as in Figure 10. In either case, the triangle T with vertices 0, z, and $\mathrm{Re}z + 0i$ is a lattice 45-45-90 triangle. Clearly T has rational (integer, in fact) legs. As P' is equilateral, the hypotenuse of T is congruent to $\overline{0w}$ and so has integral length. This contradicts Corollary 5 and completes the proof. ■

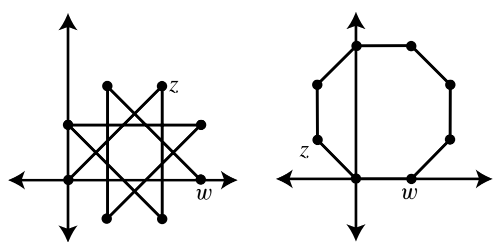

Figure 10. Lattice polygon P' with $\alpha = \pi/4$ (left) and $\alpha = 3\pi/4$ (right).

The previous two corollaries apply to angles and polygons whose vertices lie in $\mathbb{Q} \times \mathbb{Q}$. A central expansion by the least common multiple of the denominators of the coordinates of the vertices yields a similar figure with coordinates in $\mathbb{Z} \times \mathbb{Z}$, which is naturally regarded as a figure in $\mathbb{Z}[i]$.

We close this section with a seemingly unrelated application. Let X be the space obtained from the unit square $[0, 1]^2 \subset \mathbb{R}^2$ by deleting all points with both coordinates rational except (0, 0) and (1, 1). Question: can (0, 0) and (1, 1) be connected by a continuous path in X? The answer is yes, and in fact the Baire category theorem implies the existence of a smooth (infinitely differentiable) path in X from (0, 0) to (1, 1). We give an explicit example.

Corollary 8. *There is a smooth and simple path in X from* (0, 0) *to* (1, 1).

Proof. Define $\gamma : [0, 1] \to [0, 1]^2$ by $\gamma(t) = (t, (4/\pi)\arctan t)$. If the image of γ contains a rational point, then $y = (4/\pi)\arctan t$ is rational for some rational $t \in [0, 1]$. This implies $t = \tan y\pi/4$ is rational and $t = 0$ or $t = 1$ by Corollary 1. Therefore, γ is a smooth (in fact, analytic) path in X as desired. ■

The reader may enjoy producing more such paths, for example using any transcendental number $\alpha > 0$.

4. CONCLUDING REMARKS. The Gregory series for $\arctan x$ appears to have been discovered by the Indian mathematician and astronomer Mādhava in the 14th century [**17**]. It was rediscovered by Gregory in 1671 and by Leibniz in 1674 [**52**, p. 527]. Convergence of the series for $|x| < 1$ is straightforward and one employs Abel's theorem [**32**, pp. 174–175] to conclude agreement with $\arctan x$ at the endpoints $x = \pm 1$. Alternatively, one may integrate from 0 to x the identity

$$1 - t^2 + t^4 - \cdots + (-1)^{k-1} t^{2k-2} = \frac{1}{1+t^2} - \frac{(-1)^k t^{2k}}{1+t^2}$$

and note that the remainder integral tends to 0 as k tends to infinity precisely when $|x| \leq 1$.

Mādhava also discovered Leibniz's series ($x = 1$ in Gregory's series) in the 14th century and formulated correction terms for the nth partial sum [**18**]. Leibniz rediscovered the series in 1674 and published it in 1682; Gregory did not publish the series, although undoubtedly he was aware of it [**3**, pp. 132–133]. In 1730, Stirling applied transformation methods to Leibniz's series to obtain more efficient series which he used to compute $\pi/4$ correctly to 10 and 17 decimals [**47**, pp. 183–185, 223–225].

There exist other (non-Taylor) series for $\arctan x$, notably Euler's [**55**] and Castellanos's series [**11**, p. 85] (see also [**37**, p. 77]). As with Gregory's series, they too benefit convergence-wise from arguments of small absolute value.

Arctangent identities were originally discovered using (co)tangent angle addition formulas, one of which is attributed to C. L. Dodgson (Lewis Carroll) [**53**, Sec. 2]. Newton's 1676 identity in Section 1 is the earliest nontrivial arctangent identity for π known to the author and may be the first ever [**48**, p. 2].

In the 1800s, the number theory of the (later named) Gaussian integers revolutionized the search for these identities. In 1894, Dmitry A. Grave [**16**] published a problem requesting all rational integer solutions to

$$\frac{\pi}{4} = m \arctan\frac{1}{p} + n \arctan\frac{1}{q}. \tag{4}$$

Störmer took up Grave's problem, which had already been posed by Euler [**44**, p. 3], [**42**, p. 160]. Störmer solved the *a priori* more general

$$k\frac{\pi}{4} = m \arctan\frac{1}{x} + n \arctan\frac{1}{y} \tag{5}$$

over the rational integers [**43, 44, 42**]. (A gap in Störmer's proof was filled by Ljunggren in 1942 [**30**, p. 141].) Assuming that $k, m, n \geq 0$, $x \neq \pm y$, $x \neq \pm 1$, $y \neq \pm 1$, and $\gcd(m, n) = 1$, equation (5) has four solutions, namely

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}, \qquad \text{(Machin, 1706 [48, p. 7])} \tag{6}$$

$$\frac{\pi}{4} = 2\arctan\frac{1}{2} + \arctan\frac{1}{-7}, \qquad \text{(Machin, 1706 [48, p. 7])} \tag{7}$$

$$\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7}, \text{ and} \qquad \text{(Machin, 1706 [48, p. 7])} \tag{8}$$

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} + \arctan\frac{1}{-239}. \qquad \text{(Machin, 1706 [20, 48])} \tag{9}$$

Commonly, (6)–(8) are erroneously attributed to L. Euler in 1737, J. Hermann in 1706, and C. Hutton in 1776, respectively (see [**6**, p. 345], [**23**, p. 662], [**55**, p. 645], and [**11**, pp. 92, 94]). However, John Machin had already discovered (6)–(9) in 1706 [**48**], [**46**, pp. 64–66, 105–111]. Moreover, Robert Simson in 1723 had already discovered the last identity in Section 1, which is commonly attributed to Klingenstierna in 1730 [**48**, p. 10]. Indeed, Ian Tweddle's overlooked historical gem [**48**] vindicates the logical mind as it is unfathomable that (9) was discovered 31 years prior to (6). Newton's earlier 1676 identity is considerably inferior to all of Machin's formulas and was communicated without proof [**48**, p. 2]. Thus, we are led to believe that Newton was unaware of formulas (6)–(9) in 1676. Still, it would be surprising if (6) were not known prior to 1706, although the author knows no reference.

Störmer remarked that after completing his 1894 work he found that Gauss had already observed the connection between "complex integers and the arc-tangents" [**44**, p. 11]. Gauss's work is described in Schering's *Comments* section concluding Gauss's second volume of *Werke* [**15**, pp. 496–502]. Gauss used extensive tables and factorizations of "Gaussian integers" to obtain identities such as

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} \quad \text{(Gauss, 1863 [15, p. 501])}$$

which is the best three-term identity of the form (2) with $b_1 = b_2 = b_3 = 1$ [**13**]. Störmer's work was more systematic and thorough than Gauss's. In fact, Schering [**15**, pp. 499–500] states that "the developments coming from this which can be found in the written notes are not very expansive and what follows are the ones that went the farthest." For further reading on multiple-term arctangent identities see [**53, 54, 23, 12, 13**].

The fact (Corollary 2) that

$$k\pi = n \arctan \frac{b}{a} \tag{10}$$

has only the obvious rational integer solutions, namely $b = 0$ or $b/a = \pm 1$, was apparently first published by Störmer [**44**, pp. 26–27], [**42**, pp. 162–163]. His latter proof is similar in spirit to our proof of the Main Lemma (see also [**45**, pp. 166–167]). Störmer writes (10) in the form

$$\rho \arctan \frac{b}{a} = k \frac{\pi}{4} \tag{11}$$

and assumes that ρ and k are positive, $\gcd(\rho, k) = 1$, and $\gcd(a, b) = 1$. The factor of 4 in the denominator appears throughout the work of Grave, Gauss, Störmer, and others, both in single- and multi-term identities (for instance, see (4) and (5) above). This 4 causes no loss of generality, as one must verify in each proof, and is included so that the associated Gaussian integer equation has $(1 + i)^k$ on one side. However, Corollary 3 above shows very naturally that, in fact, rational arctangent identities in general may only realize integer multiples of $\pi/4$.

Another approach to Corollary 1 is to show that the only rational roots of the rational functions $\tan(k \arctan x)$, $k \in \mathbb{N}$, are $x = 0$ and $x = \pm 1$. These rational functions are the tangent analogues of the Chebyshev polynomials of the first kind for cosine. They were known to John Bernoulli as early as 1712 [**39**, pp. 193–195] and appear in Euler's 1748 work [**14**, §249] (see also [**48**, p. 8]). The author independently discovered these rational functions and proved Corollaries 1 and 2 as an undergraduate after an unsuccessful computer search for a useful single-term identity [**9**]. Several other mathematicians have used these rational functions to prove Corollary 1, notably Underwood [**49, 50, 51**] (see also Richmond [**31**]), Olmsted [**28**], which is terse but correct, and Carlitz and Thomas [**10**]. The existence of these elementary non-Gaussian integer approaches, along with Störmer's remark that Euler had posed (4) before Grave, leads us to suspect that Euler may have had a proof of Corollary 1. Using unique factorization in certain polynomial rings, one may go farther than Corollary 1 and determine the algebraic degree of $\tan k\pi/n$ over $\mathbb{Q}$ [**26**, Ch. 3].

The 20th century brought the electronic computer, the calculation of π to 100,000 decimals by Shanks and Wrench on July 29, 1961 [**37**] using arctangent identities, and, in an interesting twist, the calculation of identities themselves using the computer by Wetherfield and Chien-Lih beginning in the 1990s [**53, 12, 13**]. The calculation of π flourished in other interesting directions as well [**8, 6, 1, 7**]. For example, there exist certain formulas for computing isolated digits of π in certain bases that are powers of two and no other bases [**1, 5**]. Nevertheless, the utility of rational arctangent identities remains, as shown by Kanada's record-holding calculation of π to 1.2411 trillion decimals in December 2002 [**21**] using the self-checking pair

$$\frac{\pi}{4} = 44\arctan\frac{1}{57} + 7\arctan\frac{1}{239} - 12\arctan\frac{1}{682} + 24\arctan\frac{1}{12943} \tag{12}$$

and

$$\frac{\pi}{4} = 12\arctan\frac{1}{49} + 32\arctan\frac{1}{57} - 5\arctan\frac{1}{239} + 12\arctan\frac{1}{110443}. \tag{13}$$

Equation (12) is due to Störmer in 1896 [**44**, p. 85] and (13) is due to Kikuo Takano in 1982 [**21, 7**]. Lehmer introduced a natural measure of the efficiency of an arctangent identity for π in 1938 [**23**] which yields the compound measure of a pair of identities [**13**]. For the compound measure of the above and other self-checking pairs, see [**13**] where the Störmer-Takano pair (12)–(13) is listed under "Self-checking pairs of identities incorporating six distinct cotangent values."

Questions concerning polygons on geoboards and more general lattices have been well studied. These problems are particularly accessible to young students and admit a variety of elementary solutions. For the nonexistence of an equilateral triangle on a geoboard see [**24**], [**19**, pp. 4, 58], [**29**], [**2**, pp. 119–120], [**36**, p. 761] and [**4**, pp. 250–251]. We cannot resist presenting another solution (compare the proof of Corollary 6): suppose $v, w, z \in \mathbb{Z}[i]$ are the vertices of an equilateral triangle; translation by $-v$ shows we may assume $v = 0$; multiplication by $\overline{w}$ shows we may further assume $w \in \mathbb{Z}^+$; but then $(w/2)\sqrt{3} = \operatorname{Im} z \in \mathbb{Z}$, a contradiction.

For regular polygons on a geoboard and higher-dimensional lattices see [**34**, pp. 49–50], [**33**], [**22**], [**35**], [**4**], [**25**], and [**27**]. Note that, contrary to [**27**, bottom of p. 50], any proof for the integral lattice immediately implies the result for rational coordinates (see the paragraph following Corollary 7 above). Scherrer's 1946 infinite-descent nonexistence proof of a simple regular polygon with $n \neq 3$, 4, or 6 sides on any lattice is short and ingenious [**33**] (or see [**19**, pp. 4, 58], [**36**, pp. 761–762] or [**4**, p. 251]).

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Buel on Mathematics

"If time is of no consequence, and a fortune is already in your hands, study mathematics, because in them will be found an excellent training for the mind, but if you are a young man compelled to make your own way in the world, and particularly if you are not endowed with the mathematical gift, it is altogether probable that you will find sawing wood more profitable than studying these top-lofty branches, where the fruit is extremely difficult to pluck, and often of poor flavor, especially when gathered by a man who has neither craft nor profession."

J. W. Buel, *Buel's Manual of Self Help*,
National Book Concern, Chicago, 1894, p. 69.

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