Problems for Math 342: The Mathematics of Social Choice, Fall 2017

1. Suppose we have some number n of voters, each ranking three candidates x, y, and z.
   (a) The *instant runoff* assigns a Social Ranking as follows: the candidate with the fewest number of first place votes is third (for simplicity, assume there are no ties), and that candidate is eliminated. We eliminate that candidate from everyone's individual ranking, and recalculate first place votes; the winner is the one with the most votes. For example, if candidate y has the fewest 1st place votes, then everyone who has y as their first choice transfers their vote to their second choice, and we have a majority election between x and z.
      i. Does the instant runoff social ranking have the Pareto property (if every voter prefers x to y, then society prefers x to y)? Briefly prove or give a counterexample. Hint: To create counterexamples, it is often helpful to list some possible rankings (like $x > y > z$) and how many people vote with that ranking.
      ii. Does the instant runoff have Independence of Irrelevant Alternatives (if z moves up or down in some voters' rankings but x and y keep the same relative order to each other in each voters' ranking, then x and y keep the same relative order in the social ranking)? Briefly prove or give a counterexample.
   (b) The *antiplurality* voting system chooses the least hated candidate, as follows: the Social Ranking ranks people higher who have fewer third place votes (again, assume no ties).
      i. Does the antiplurality social ranking have the Pareto property? Briefly prove or give a counterexample.
      ii. Does the antiplurality vote have Independence of Irrelevant Alternatives? Briefly prove or give a counterexample.

2. Suppose we have some number n of voters, each ranking three candidates x, y, and z.
   (a) Give an example of a preference profile (the collection of rankings for all n voters), for an n of your choice, such that:
      - Candidate x is the sole first choice under the plurality vote (defined in class),
      - Candidate y is the sole first choice under the Borda count (defined in class), and
      - Candidate z is the sole first choice under the instant runoff (from the previous problem).
   (b) Looking at your preference profile from part (a), who do you think "should" win the vote, and why? (There's no right answer here.)

3. In an auction, each agent i has a (perhaps secret) valuation, v_i, for an object. The agent's utility from the result of the auction is 0 if they don't get the object and $v_i - p$ if they get it and have to pay p. In the second-price auction with bids $b_1, \dots, b_n$ the agent with the highest bid wins the object (for simplicity, assume there are no ties) and pays the second highest bid. Prove that, no matter what $b_2, \dots, b_n$ are, agent 1 can have no

utility higher than she would get if she bids exactly v_1. Please prove this carefully; I suggest some precisely-stated cases. Hint: think of the $b_2, \dots, b_n$ as fixed, and consider what utility agent 1 might get from various bids b_1. (Of course, the statement holds for all agents, not just agent 1, so it is in everyone's best interest to bid their valuation, *no matter what* other people bid).

4. Assume three candidates, B(ush), G(ore), and N(ader) are in an election (on the political spectrum, G is in between the other two). Assume an agent's preferences are completely based on where she is on the political spectrum. In particular, anybody who has N as her first choice must be more left-wing than G, and so must be way more left-wing than B, and therefore she must have G as her second choice and B as her third; similarly, anybody who has B as her first choice must have G as her second and N as her third. Assume there are an odd number of agents (this is to prevent ties).
   (a) Assume people vote their true preferences (in particular, no one votes N>B>G or B>N>G). Prove that there is always a Condorcet winner (an alternative that wins a head-to-head majority election against any other opponent).
   (b) Suppose people only vote for their first choice (so you don't get to see who their second or third choices are). Design a system, using only that information, for figuring out who the Condorcet winner is (and prove it works). You are allowed to use particularities of the situation, e.g., that G is between B and N on the spectrum.
   (c) Is your voting system in part (b) strategy-proof? That is, does any agent ever have an incentive to lie about their first choice? Prove or give a counterexample.

5. Suppose we have some number, n, of voters choosing one of three candidates from $X = \{x, y, z\}$.
   (a) Give an example of a preference profile (set of votes), for an n of your choice, such that the following all hold simultaneously:
      - Candidate x is the sole first choice under the plurality vote,
      - Candidate y is the sole first choice under the Borda count, and
      - Candidate z is the Condorcet winner (wins head-to-head against x and wins head-to-head against y).

      Two bonus points for whoever can do it with the smallest number of voters, n.
   (b) Looking at your preference profile from part (a), who do you think "should" win the vote, and why? Write a paragraph or so defending your choice.

6. For this question, it is more convenient to think about the Reduced-Borda count, which is equivalent to the Borda count: If there are k candidates and n voters, a voter's first place choice gets $k-1$ points, second place gets $k-2$, ..., kth place gets 0; this is one less than their normal Borda points, for each voter (so each candidate will get n fewer points in all). The winner is the person with the highest total Reduced-Borda count, and the winner is the same as in the normal Borda count.
   (a) Prove that the average Reduced-Borda score among the k candidates is $n(k-1)/2$.

(b) For a voter i and candidates x and y, define

$$P_i(x,y) = \begin{cases} 1 & \text{if voter } i \text{ prefers } x \text{ to } y \\ 0 & \text{otherwise} \end{cases}$$

Give an equation relating a candidate a's Reduced-Borda count to these $P_i(x,y)$ functions (you'll use various i, x, and y in the formula).

(c) Suppose that candidate a is the Condorcet winner. Prove that their Reduced-Borda count is larger than the average, $n(k-1)/2$.

(d) Suppose we use the following instant runoff voting system:

- Calculate the Reduced-Borda counts of all of the candidates.
- Remove the candidate who has the smallest count (for simplicity, assume there are never ties).
- Recalculate all of the Reduced-Borda counts for the remaining $k-1$ candidates (e.g. if voter 1 had $b >_1 c >_1 a$ originally and c is removed, their new preference is $b >_1 a$).
- Again, remove the candidate with the smallest count.
- Continue until there is only one candidate remaining.

Prove that, if there is a Condorcet winner, then they win under this Borda-Instant-Runoff system.

7. We say that a Social Choice function (electing a single winner) is *monotone* if, whenever a candidate x is currently winning, x still wins if some voter(s) move x up in their rankings but leave the relative order of the other candidates the same (e.g., if a voter has $y > z > x$ they can change their vote to $x > y > z$ but not to $x > z > y$). For each of the following election systems, either *briefly* prove that it is monotone or provide a counterexample.

   (a) Plurality.

   (b) Borda.

   (c) Instant Runoff.

8. Suppose we have n agents, numbered 1 through n, and k identical cookies, with $k < n$. We will create an auction where each agent gets at most one cookie. Suppose that θ_i is agent i's valuation for getting a cookie. Assume that all θ_i are distinct and positive (this prevents ties). Given a subset $S \subseteq \{1, \ldots, n\}$ with $|S| \leq k$, let d_S be the decision where each agent in S gets a cookie (we allow $|S| < k$ in case not all of the cookies are given out). Let $D = \{d_S : S \subseteq \{1, \ldots, n\}, |S| \leq k\}$ be the set off all of these possible decisions.

   To clarify notation, I suggest assuming (without loss of generality) that $\theta_1 > \theta_2 > \cdots > \theta_n$. We discussed in class that a reasonable auction might give a cookie to agents 1 through k and charge them θ_{k+1}. We *carefully* show here that this is what the Clarke Pivot mechanism does, following the notation from Lecture 7:

(a) For each agent i and decision d_S, give a formula for $v_i(d_S, \theta_i)$.

(b) Carefully compute the efficient decision function $d(\Theta)$.

(c) For each agent i, carefully compute the $t_i(\Theta)$ given by the Clarke Pivot mechanism.

9. If $>_1$ and $>_2$ are two rankings of candidates, we define the *Kendall-tau distance* between the rankings to be the number of pairs of candidates such that $>_1$ and $>_2$ disagree about that pair, that is,

$$d(>_1, >_2) = \#\text{ pairs } \{a, b\} : \ (a >_1 b \text{ and } b >_2 a) \text{ or } (b >_1 a \text{ and } a >_2 b).$$

For example, $d(a >_1 b >_1 c,\ c >_2 b >_2 a) = 3$, because the voters disagree about their preferences for each pair of candidates: $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. If $>$ is a ranking, and $\mathcal{V}$ is a set of rankings (think of $>$ as a possible social ranking and $\mathcal{V}$ as the set of votes), then define

$$d(>, \mathcal{V}) = \sum_{>_i \in \mathcal{V}} d(>, >_i).$$

We want the social ranking we decide on to be as close as possible to the set of votes, so given a set of votes $\mathcal{V}$, define the *Kemeny-Young social ranking* to be the ranking $>$ minimizing $d(>, \mathcal{V})$. Note this selects a full ranking of the candidates; you can simply define the winner as the first place in the ranking.

(a) We say that a social ranking system is *consistent* if, whenever one set of voters $\mathcal{V}$ selects a social ranking $>$ and a disjoint set of voters $\mathcal{V}'$ selects the same ranking $>$, then combining the votes together into $\mathcal{V} \cup \mathcal{V}'$ still selects $>$. Prove that the Kemeny-Young social ranking is consistent.

(b) Suppose that $>$ and $>'$ differ only by switching two adjacent candidates (call them a_j and a_{j+1}), that is

$$a_1 > a_2 > \cdots > a_k \quad \text{and}$$

$$a_1 >' a_2 >' \cdots >' a_{j-1} >' a_{j+1} >' a_j >' a_{j+2} >' a_{j+3} >' \cdots >' a_k.$$

i. Given a single vote $>_i$, how do $d(>, >_i)$ and $d(>', >_i)$ differ?

ii. Given a set of votes $\mathcal{V}$, how do $d(>, \mathcal{V})$ and $d(>', \mathcal{V})$ differ?

(c) Prove that, if there is a Condorcet winner, then the top candidate in the Kemeny-Young social ranking is the Condorcet winner.

(d) Philosophical Question: One can prove that the Kemeny-Young social ranking is the unique system that is consistent, elects the Condorcet winner (when there is one), and treats all voters equally (symmetrically). What do you think of that?

10. (10 points) Consider the following indirect mechanism for auctioning off k identical cookies, in the situation where people may want more than one cookie (and each person has decreasing marginal utility, as in the example in Lecture 8). We start at price $p = 0$ and slowly raise the price; person i holds up k_i fingers, if they would want to buy k_i cookies each at price p, but not k_{i+1}, that is

$$\text{marginal utility of } k_i\text{th cookie} > \ p \ > \text{marginal utility of } k_{i+1}\text{st}).$$

Assuming that $\sum_i k_i > k$, we continue to slowly raise the price. The k_i's will decrease as p rises, and eventually, we will have $\sum_i k_i = k$ exactly (assume there are no ties in valuations, so the sum will never decrease by 2 or more at once). Then person i gets k_i of the cookies, and they all pay p per cookie.

(a) This indirect mechanism seems like the most likely auction you would come up with if you were designing one. It is not equivalent to the Ascending Price mechanism from class, because here all of the cookies go for the same price, and we saw that wasn't true for the example in class. Use some of our big Theorems from Lectures 9 and 10 to show that this new mechanism cannot be strategy-proof.

(b) Using $k = 4$ and the example types from class

$$\theta_I = (10, 8, 6, 1), \quad \theta_J = (9, 7, 2, 0), \quad \theta_K = (5, 4, 3, 0),$$

explicitly show how someone might strategize to get a better outcome.

11. (15 points) This problem goes on forever (or at least onto the next page). You and your roommate, Sam, are deciding whether to buy a fancy coffee maker for your house/room. It cost 100 dollars (fancy!). Each of you have a (perhaps secret) amount of your own money that you are willing to pay to get the machine. Call these amounts θ_1 and θ_2 for you and Sam, respectively. For example, if $\theta_1 = \theta_2 = 60$, then you could both pay \$50, and you would each gain $60 - 50 = \$10$ in utility.

(a) Suppose you are willing to pay \$40 and Sam is willing to pay \$80. Should you get the machine? Philosophical question: what's a "fair" price for each of you to pay?

(b) Suppose you agree on the following mechanism: you simultaneously reveal how much you are willing to pay, say b_1 and b_2 (for you and Sam, respectively). If $b_1 + b_2 \geq 100$, you buy the machine, you yourself contribute $p_1 = 100b_1/(b_1 + b_2)$, and Sam contributes $p_2 = 100b_2/(b_1 + b_2)$. Nothing stops you from lying (having $b_1 \neq \theta_1$). Note: $p_1 + p_2 = 100$, so this exactly pays for the machine.

Is this mechanism strategy-proof? That is, no matter what θ_1 and b_2 are, is bidding $b_1 = \theta_1$ at least as good as any alternative bid? To be clear, if you don't buy the machine, then your utility is 0, and if you do buy it, then your utility is $\theta_1 - p_1$.

Next let's translate this problem into the language of the VCG mechanism. To do this, we need to internalize the cost of the machine: assume that, by default, the \$100 cost will be split evenly (transfers can alter this baseline), so

$$v_i(Buy, \theta_i) = \theta_i - 50.$$

That is, losing that \$50 is considered a part of the decision, separate from any transfers that may happen. On the other hand,

$$v_i(No\ Buy, \theta_i) = 0,$$

because this is the baseline zero utility of nothing happening. Let $d(\Theta)$ be the *efficient* decision function. Be careful in these problems to do the calculations based on the formulas for v_i, rather than your intuition.

(c) What is $d(\Theta)$, as a function of $\Theta = (\theta_1, \theta_2)$?

(d) Let's try a VCG mechanism... we just have to figure out which one is most reasonable. You could try the Clarke Pivot mechanism, though you would find that it is not individually rational (you might be forced to pay more than you are willing to, when buying the coffee maker). Instead, let's specify particular $h_i(\theta_{-i})$, rather than the one specified by the pivot mechanism. Let's take the VCG mechanism that has

$$h_1(\theta_2) = h_2(\theta_1) = 0.$$

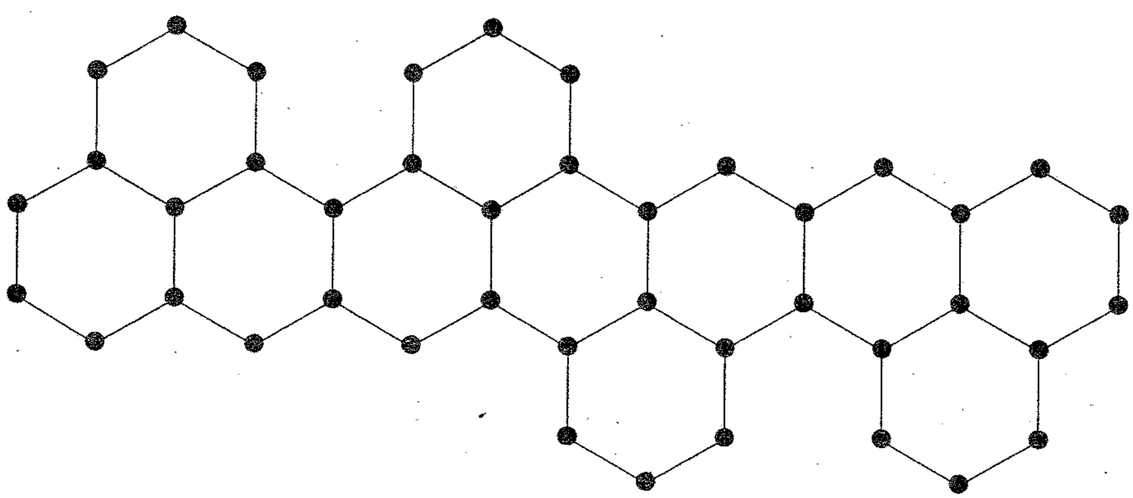
i. As a function of Θ compute $t_1(\Theta)$ and your net utility $t_1(\Theta) + v_1\big(d(\Theta), \theta_1\big)$,

ii. Prove this mechanism is individually rational.

iii. Examine the case where $\theta_1 = 40$ and $\theta_2 = 80$. What are t_1 and t_2? Including the \$50 that we assumed each payed by default, how much do each pay? Individual rationality holds, but what's the problem now?

(e) Philosophical question: suppose y'all have a kitty of \$200 already set aside for household expenses, but still run the mechanism in part (d) and contribute money based on it; the kitty simply pays the shortfall, if there is any. What a great way to handle this problem of a possible shortfall! You yourself *really, really* want a stand mixer to bake cookies for your Math 342 class, and y'all might run a similar mechanism in a few days to decide whether to buy it. Does this affect your strategy?

12. (5 points) Either demonstrate a perfect matching (every vertex gets matched to some other one) in the following graph or prove that no such matching exists. Hint: This is a bipartite graph, though not drawn that way. From class, we've seen that a proof that no such matching exists can be short and sweet (and we've even seen an algorithm for finding that proof)! I'm giving you a bunch of copies of this graph at the end of the problem set, to play around with.



13. (a) We're in the matching with money context. Suppose we have n buyers and n items to sell, and suppose every buyer agrees on the valuations of each item (but each item is different). That is, for each item j, there is some v_j such that $v_{ij} = v_j$ for all buyers i.

    i. Describe the steps the Ascending Price mechanism will go through, and the final payments, p_j.

    ii. Describe the set of all prices $(p_1, \ldots, p_n)$ that are stable.

    (b) We're again in the matching with money context. Suppose we have $k < n$ *identical* items to sell (but, in contrast to part (a), the buyers disagree about their valuation), and $n - k$ dummy items valued at 0 by everyone. That is, for each buyer i, there is some v_i such that $v_{ij} = v_i$ for $1 \leq j \leq k$, and $v_{ij} = 0$ for $k + 1 \leq j \leq n$.

    i. Describe the steps the Ascending Price mechanism will go through, and the final payments, p_j.

    ii. Where have we seen this outcome before?

    iii. Describe the set of all prices $(p_1, \ldots, p_n)$ that are stable.

14. We're in the matching with strict preferences, but *no money*, context. Assume we have 3 men (A,B,C) and 3 women (R,S,T).

    (a) Give an example of a set of preferences for which there is exactly 1 stable matching. Hint: The shorter the preference lists (the more people who are unacceptable), the easier it is to prove stability.

    (b) Give an example for which there are exactly 2 stable matchings.

    (c) Give an example for which there are exactly 3 stable matchings.

    (d) Bonus: I think there cannot be more than 3 stable matchings in this situation, but don't have a simple proof. Can you prove or disprove it?

15. We're in the matching with money context, and the free-for-all auction described on page 75 of the notes: The first person to bid on any cookie bids 0; any buyer who is not currently winning a cookie may bid ε over the current bid for a cookie; and bidding continues until no one wants to bid. Assume that a player will always bid on a cookie that maximizes their net utility, at current prices.

    (a) Prove that, until the very end of the auction, there are still cookies that have not been bid on (Hint: if m cookies have been bid on so far, how many buyers are allowed to make a bid?). Therefore the last cookie to be bid on (perhaps a dummy one) will go for free, and we can assume that everyone ends up with a cookie (because anyone is willing to take a free cookie).

Let q be the price vector at some point in the auction, with corresponding matching μ_q (this might not yet be a perfect matching). Let p be the price vector from the Pivot rule / Ascending price mechanism that we have been discussing in class, with corresponding perfect matching μ_p. Let k be the number of non-dummy cookies (the only ones that could possibly go for positive amounts). We say that buyer i's *unhappiness level*, u_i, is how much worse off they are with (q, μ_q) as opposed to (p, μ_p); that is

$$u_i = (v_{i\mu_p(i)} - p_{\mu_p(i)}) - (v_{i\mu_q(i)} - q_{\mu_q(i)}).$$

We say that a cookie j is *overpriced* by the amount c if $q_j - p_j = c$. We want to prove that no cookie will ever be $> 2k\varepsilon$ overpriced (the proof that no cookie will end up $> 2k\varepsilon$ underpriced is similar), and therefore this free-for-all auction gives a similar result to the more complicated version.

(b) Suppose a cookie j is c overpriced, with $\mu_q(i) = j$. Prove that

  i. i has unhappiness level $\geq c$.

  ii. $\mu_p(i)$ is $\geq c - \varepsilon$ overpriced. Hint: when buyer i bid on cookie $\mu_q(i)$, that was the best bid they could make, at the time.

(c) Seeking a contradiction, assume some cookie j is $> 2k\varepsilon$ overpriced.

  i. Find 2 cookies that are each $> 2(k-1)\varepsilon$ overpriced, either now or as the auction continues. Hint: You'll need 2 cases, depending on whether the buyer i with $\mu_q(i) = j$ also has $\mu_p(i) = j$ or not.

  ii. Find 3 cookies that will now or shortly each be $> 2(k-2)\varepsilon$ overpriced.

  iii. Continuing this process, find a contradiction.

16. We're in the matching with strict preferences, no money, context.

  (a) Prove that the set of men and women who are matched is the same for any stable matching. Hint: Let μ_M be the matching in the men-propose mechanism, and let μ be another stable matching. Let M_1 and M_2 be the set of men who are matched in μ_M and μ, respectively. What can you more easily prove about how M_1 and M_2 are related? What about for the women matched in μ_M and μ?

  (b) Let f be a stable matching mechanism. That is, f is a function mapping a preference profile (list of everyone's preferences) to a matching that is stable for that profile (and that is all we are given about the output of f). Let P be the true preference profile, and suppose $f(P) = \mu$. If $\mu \neq \mu_W$, prove that some woman has the incentive to lie about her preferences, that is, if she lies about her preferences and creates a new preference profile P', she can lie in such a way that she prefers $f(P')$ to $f(P)$. Hint: one way to lie would be to truncate her preferences (say that the men below a certain point on her list are unacceptable when they are, in fact, acceptable).

17. We're in the matching with strict preferences, no money, context. Use any number of men and women you'd like, though I suggest 4 each.

   (a) Give an example of preferences, where there are at least four distinct stable matchings, μ_W, μ_1, μ_2, and μ_M, where some women prefer μ_1 to μ_2 and some women prefer μ_2 to μ_1.

   (b) Give an example of preferences, where there are at least four distinct stable matchings, μ_W, μ_1, μ_2, and μ_M, where every woman is at least as happy in μ_1 as μ_2.

   Hint: I've found that limiting (as much as possible) the number of people that people find acceptable to be helpful in limiting the amount of craziness.

18. We are in the matching with strict preferences, no money, context.

   (a) Let P be a set of preferences, with n men $m_1, \ldots, m_n$ and n women $w_1, \ldots, w_n$. Assume no one finds anyone unacceptable. Let's look at $2n$ men $m_{11}, \ldots, m_{1n}, m_{21}, \ldots, m_{2n}$ and $2n$ women $w_{11}, \ldots, w_{1n}, w_{21}, \ldots, w_{2n}$, and create preferences P' among them. First we'll state the preferences precisely, and then a little more intuitively. Say that m_{ij} prefers $w_{k\ell}$ to w_{rs} if either:

   - $k = r$ and m_j prefers w_ℓ to w_s in P, or
   - $k = i$ and $r \neq i$,

   and w_{ij} prefers $m_{k\ell}$ to m_{rs} if either:

   - $k = r$ and w_j prefers m_ℓ to m_s in P, or
   - $k \neq i$ and $r = i$.

   That is,

   - Every m_{ij} has the same preferences on the $w_{1\ell}$ as m_j did on the w_ℓ, and the same preferences on the $w_{2\ell}$ as m_j did on the w_ℓ,
   - Similarly for the women,
   - Every m_{1j} prefers all of the $w_{1\ell}$ to all of the w_{2s}, and every m_{2j} prefers all of the $w_{2\ell}$ to all of the w_{1s}, and
   - Every w_{1j} prefers all of the $m_{2\ell}$ to all of the m_{1s}, and every w_{2j} prefers all of the $m_{1\ell}$ to all of the m_{2s}.

   Suppose there are N stable matchings under P. Construct $2N^2$ stable matchings under P' (you only need briefly prove they are stable).

   (b) Prove that, for all $n = 2^k$ with k a nonnegative integer, there is some preference ordering P for n men and n women with at least 2^{n-1} stable matchings. That's a lot of stable matchings!

19. Let's examine the first-price auction with n agents, generalizing what we did with $n = 2$ agents on Friday 10/27. Assume that agent i values the object at some Θ_i chosen uniformly at random from $[0, 1]$, and that each agent's valuation is independent.

(a) Suppose agents 2 through n follow the strategy of bidding exactly their valuation Θ_i. Compute what bid $b(\theta_1)$, as a function of θ_1, agent 1 should make to maximize his expected utility, when his valuation is $\Theta_1 = \theta_1$. Hint: the proof for $n = 2$ can be followed fairly closely.

(b) Assume agents 2 through n follow the bidding strategy $b(\Theta_i)$, where b is the bidding function you computed in part (a). Prove that agent 1 should also bid $b(\theta_1)$, when $\Theta_1 = \theta_1$, in order to maximize his expected utility. (That is, it is a Bayesian Nash Equilibrium.)

20. Suppose we have an auction with n agents, where (as in problem 1) agent i values the object at some Θ_i chosen uniformly at random from $[0, 1]$, and that each agent's valuation is independent. For each of the following random variables X, compute the cumulative distribution function $F(x) = P(X \leq x)$ and the density function $f(x) = F'(x)$.

    (a) $X = \max_i(\Theta_i)$.

    (b) $X = \min_i(\Theta_i)$. Hint: first compute $P(X > x)$.

    (c) $X =$ the second highest value from among the Θ_i.

21. Suppose we again have an auction with n agents, where agent i values the object at some Θ_i chosen uniformly at random from $[0, 1]$, and that each agent's valuation is independent. We will compute the expected revenue for the seller in both the first and second price auction.

    (a) For the first price auction, assume every agent bids following the (Bayesian Nash Equilibrium) bidding strategy $b(\cdot)$ you calculated in problem 1. Calculate the expected revenue for the seller in this case.

    (b) For the second price auction, assume every agent follows their dominant strategy and bids exactly their valuation Θ_i. Calculate the expected revenue for the seller.

22. Assume we have 2 bidders with respective valuations Θ_1 and Θ_2 for a cookie, and Θ_1 and Θ_2 are i.i.d. uniform $[0, 1]$. Suppose the auctioneer announces it will be a second price auction with reserve price r (r is known to everyone at the time of bidding). That is, if neither agent bids at least r, neither gets the cookie; otherwise, the highest bidder gets the cookie and pays the greater of r or the second-highest bid.

    (a) Briefly show that bidding your true valuation is strategy-proof.

    (b) Compute $m(\theta_1)$, the (interim) expected payment for Agent 1, if $\Theta_1 = \theta_1$ (but Θ_2 is unknown); this will be a function of θ_1 and r. (Careful: your final answer should be a piecewise defined function of θ_1, and that will be important in part (c)).

    (c) Compute the (ex ante) expected revenue for the auctioneer; this will be a function of r only.

    (d) What reserve r should the auctioneer set in order to maximize (ex ante) expected revenue?

23. Assume we have 2 bidders with valuations i.i.d. uniformly at random from $[0, 1]$. Suppose the auctioneer announces it will be a *first* price auction with reserve price r (r is known to everyone at the time of bidding). That is, if neither agent bids at least r, neither gets the object; otherwise, the highest bidder gets the object and pays their bid. Consider the bidding strategy,

$$b(\Theta_i) = \begin{cases} 0 & \text{if } \Theta_i < r, \\ \frac{r^2+\Theta_i^2}{2\Theta_i}. & \text{if } \Theta_i \geq r \end{cases}$$

(a) Prove that $b(\theta)$ is increasing, for $r \leq \theta \leq 1$,

(b) Prove that $b(\cdot)$ is a Bayesian Nash Equilibrium. Hint: I think it's easiest to use the trick from Unit 24 where you write a potential bid b^* as $b^* = b(\psi)$, for some ψ.

(c) Given $\Theta_1 = \theta_1$ and both agents are following the above strategy, compute $m(\theta)$, the interim expected payment of Agent 1. If you did this and Problem 1(b) correctly, you'll notice a suspicious similarity.

(d) Compute the (ex ante) expected revenue of the seller, when the buyers follow this strategy. If you did this and Problem 1(c) correctly, this might feel repetitive.

24. Coffee makers, aka, public goods (if bought, everyone gets the utility of using it). Don't forget that you can use WolframAlpha/Mathematica. Suppose two roommates have respective valuations Θ_1 and Θ_2 for an object, and Θ_1 and Θ_2 are i.i.d. uniform $[0, 1]$.

Suppose the object costs \$1. If $\Theta_1 + \Theta_2 \geq 1$ and they manage to buy it, then the total surplus is $\Theta_1 + \Theta_2 - 1$; otherwise it is 0. I suggest paying careful attention to Units 25 and 26 when solving this problem.

(a) Suppose they somehow manage to truthfully state their valuations and buy the object whenever $\Theta_1 + \Theta_2 \geq 1$. What is the (ex ante) expected total surplus? (Of course, there is no strategy-proof way to ensure this is achieved.) Hint: you may want to first compute an interim expected surplus.

(b) Suppose we use the following mechanism: if they both say they value it for at least 0.5, they buy it and each pay 0.5. This is a strategy-proof mechanism. What is the (ex ante) expected total surplus?

(c) Suppose we use the following mechanism: The roommates simultaneously name prices b_1 and b_2, respectively. If $b_1 + b_2 \geq 1$, then they buy it; roommate 1 pays

$$b_1 - \frac{b_1 + b_2 - 1}{2} = \frac{b_1 - b_2 + 1}{2},$$

and roommate 2 pays $(b_2 - b_1 + 1)/2$; that is, they start by paying b_1 and b_2, respectively, but each get back half of the extra money that this generates.

Consider the symmetric, linear bidding strategies $b_i(\Theta_i) = \frac{2}{3}\Theta_i + \frac{1}{12}$.

i. Prove that these strategies are a Bayesian Nash Equilibrium.

ii. Suppose both roommates follow this strategy. Compute the (ex ante) total expected surplus.

(d) How do your three (ex ante) expected surpluses for these three different mechanisms compare?

These next six questions are philosophical questions. Please be sure you're doing some minimal about of mathematics somewhere here, though! Come to class prepared to discus

25. Alexis and Brenda are housemates. One day, they find a carton of 12 grapefruit and 12 avocados on their front porch. They need to divvy them up. Scientists are just now realizing the crucial nature of vitamin F for your health, and grapefruit and avocados are great sources of it. Alexis and Brenda's health could really benefit from getting some vitamin F. Both Alexis and Brenda had recently been to the doctor, who can do amazing tests nowadays to determine your body's ability to derive vitamin F from food. They learned:

    - Alexis can derive 100 mg of vitamin F from each grapefruit consumed, but no vitamin F from avocado.
    - Brenda can derive 50 mg of vitamin F from each grapefruit and 50 mg from each avocado.

    These stats are common knowledge to Alexis and Brenda. Both Alexis and Brenda are strongly committed to getting as much vitamin F as they can. Other than that, neither of them particularly like or dislike avocados and grapefruit.

    (a) Assume Alexis and Brenda are good friends who want to do the "fair" thing when dividing up the fruit carton. How should they do it?
    (b) Assume Alexis and Brenda don't really care for one another, but only care for how much vitamin F they get individually. If they can't agree on how to divvy up the grapefruit and avocado, the fruit will sit there until they spoil. How do you think they will end up splitting the carton?

26. Suppose that, instead of deriving 50 mg each from grapefruit and avocado, Brenda can only derive 9.09 mg each. Repeat parts (a) and (b) under this new situation.

27. Alexis and Brenda decided it was less combative to just put the carton on the doorstep of their next door neighbors, Carla and Deb, and see what they did with this free gift. Carla and Deb don't go in for any of this vitamin F crap. However, they do like grapefruit and avocados, to varying degrees:

    - Carla is quite fond of grapefruit and would pay up to \$1 per grapefruit for as many as you would sell her. She doesn't like avocado and wouldn't buy them at any price.
    - Deb likes both grapefruit and avocado, and would pay up to \$.50 each for as many as you would sell her.

    These stats are common knowledge to Carla and Deb. They also don't want to pay each other money (simply divide up the fruit), because money is gauche. Repeat parts (a) and (b) under this new situation.

28. Suppose that, instead of feeling grapefruit and avocado are worth \$.50 each, Deb only feels that are worth \$.09 each. Repeat the previous parts (a) and (b) under this new situation.

29. Carla and Deb also give up on this task, and pass the grapefruits along to Edith and Felicity next door. Edith and Felicity are hard core economists with no compunction against paying each other money for anything. They have the same values as the original Carla and Deb:

    - Edith is quite fond of grapefruit and would pay up to \$1 per grapefruit for as many as you would sell her. She doesn't like avocado and wouldn't buy them at any price.
    - Felicity likes both grapefruit and avocado, and would pay up to \$.50 each for as many as you would sell her.

    These stats are common knowledge to Edith and Felicity. Repeat parts (a) and (b) under this new situation.

30. In a given auction, let $m(\theta)$ be the *ex interim* expected payment of a player with valuation $\Theta_i = \theta$ (the player knows their own valuation θ but does not yet know the other players valuations) when all players are using some equilibrium betting strategy $b(\theta)$. We proved in class that, if the auction satisfies certain assumptions (particularly, the highest bidder gets the object), then

    $$m(\theta) = \int_0^\theta g(t)t\,dt,$$

    where $g(t)$ is the density function of $\max\{\Theta_2, \ldots, \Theta_n\}$ (the max valuation of everyone else).

    Assume we are in an *all pay* auction with n players, each with valuations independently and uniformly at random from $[0, 1]$. That is, each player must pay their bid, but only the highest bidder wins the object. Let $b(\theta)$ be a symmetric equilibrium bidding strategy.

    (a) Find an (easy!) equation relating $b(\theta)$ and $m(\theta)$.

    (b) Use the previous part to get an explicit formula for $b(\theta)$.

    (c) Confirm that your answer is indeed a Bayesian Nash Equilibrium.

31. Let q_i be the *standard* quota for State i in an apportionment problem, that is,

    $$q_i = \frac{p_i}{d_s} = \frac{p_i}{p/h} = \frac{p_i}{p}h.$$

    We say that an apportionment method has the *lower quota property* if it is always true that, in the final apportionment,

    $$a_i \geq \lfloor q_i \rfloor .$$

    We say that an apportionment method has the *upper quota property* if it is always true that, in the final apportionment,

    $$a_i \leq \lceil q_i \rceil .$$

(a) Give an example to show that Webster's method does not satisfy the lower quota property. Hint: I found it helpful to make an excel spreadsheet so that I could play with numbers without having to redo all of the calculations every time.

(b) Give an example to show that Webster's method does not satisfy the upper quota property.

(c) Show that Jefferson's method satisfies the lower quota property.

32. Consider the following allocation method. We start with $a_i = 0$ for all i, that is, none of the seats are yet allocated. We are going to allocate one seat at a time. To allocate the first seat, let i be the state that maximizes $p_i/(a_i+1)$ across all states; give the seat to state i, that is, increment a_i by 1 (from 0 to 1, in this first example). To allocate the second seat, again look at all of the states, take the state j maximizing $p_j/(a_j+1)$, and increment a_j by 1. Continue, seat by seat, until all h seats have been allocated. This seems reasonable to do: the bigger $p_i/(a_i+1)$, the more of a claim state i has to the "next" seat.

    (a) Show that the final allocations a_i are exactly those of Jefferson's method. Hint: Start by analyzing Jefferson's method like we did for others in Unit 31. Then try to prove by contradiction.

    (b) How would you modify this method to give you Webster's method?

    FYI, Balinski and Young modified this version of Jefferson's method to avoid upper quota violations: simply don't allow a state to get a new seat once $a_i = \lceil q_i \rceil$, where q_i is the standard quota. Like Jefferson's original method, this still has the lower quota property, so it in fact has the (entire) quota property. It's also house monotone, because of the way it allocates seats one at a time. We have not seen any other methods that are both house monotone and satisfy the quota property.

33. Read the "Social Justice" chapter of *The Theory of Choice: A Critical Guide*, Hargreaves Heap, et al., eds. I have posted this on Blackboard. Come on Friday 12/8 prepared to discuss it. Think about how it relates to the stuff from the last few days. Come to class having thought about it.

34. Write a 300-450 word essay (1 or so pages, double spaced) relating the "Social Justice" chapter to something we've done this semester (e.g., voting, auctions, matching mechanisms, Nash bargaining,...). Don't ramble, but rather have a specific thesis, backed up with evidence (including some evidence of a mathematical nature). Because of the limited space, your thesis will, by necessity, be limited; that's fine. Your audience is another member of this class who has been engaging in the same material.

35. What's been your favorite topic in this class? Your least favorite?