

QUASIPOLYNOMIAL BEHAVIOR VIA CONSTRUCTIBILITY IN MULTIGRADED ALGEBRA

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ABSTRACT. Piecewise quasipolynomial growth of Presburger counting functions combines with tame persistent homology module theory to conclude piecewise quasipolynomial behavior of constructible families of finely graded modules over constructible commutative semigroup rings. Functorial preservation of constructibility for families under local cohomology, Tor, and Ext yield piecewise quasipolynomial, quasilinear, or quasiconstant growth statements for length of local cohomology, a -invariants, regularity, depth; length of Tor and Betti numbers; length of Ext and Bass numbers; associated primes via v -invariants; and extended degrees, including the usual degree, Hilbert–Samuel multiplicity, arithmetic degree, and homological degree.

CONTENTS

1. Introduction	2
2. Presburger arithmetic	7
2.1. Presburger definability	7
2.2. Decomposing Presburger sets	8
2.3. Semisimple atoms	9
3. Numerics of constructibility	10
3.1. Numerical constructibility	10
3.2. Quasipolynomials from numerical constructibility	11
4. Constructible modules	12
5. Flat complexes of modules	15
5.1. Homomorphisms and complexes	15
5.2. Localization	16
5.3. $\mathfrak{X}$ -flat complexes	17
5.4. Faces and localizations along them	18
6. Functors on class $\mathfrak{X}$ modules	19
6.1. Local cohomology	19

Date: 23 December 2025. The authors gratefully acknowledge the American Institute of Mathematics (AIM), where much of this research was carried out, and the Vietnam Institute for Advanced Study in Mathematics (VIASM), where N. V. Trung and L. T. Hoa hosted EM for a productive week of development. EM was partly supported by NSF DMS-2515765. JM was partly supported by NSF DMS-2401522. Thanks to Lukas Waas for finding an error in an earlier proof of Proposition 7.29.

2020 *Mathematics Subject Classification.* Primary: 13D07, 13A30, 13D45, 13L05, 13A02, 05E40, 52B20, 52C07, 20M14, 03F30, 03C10.

6.2. Tensor products and Tor	19
6.3. Hom and Ext	20
7. Constructible families	24
7.1. Rees monoids	24
7.2. Constructible families	24
7.3. Quasipolynomiality from constructibility	27
7.4. Constructible families of ideals	28
7.5. Products of ideals and modules	32
8. Functors preserving constructibility of families	34
8.1. Local cohomology	34
8.2. Tensor products and Tor	34
8.3. Hom and Ext	36
8.4. Functorial quasipolynomiality	38
9. Ubiquity of quasipolynomiality	39
9.1. Local cohomology, $\mathbf{a}$ -invariants, and regularity	40
9.2. Tor and Betti numbers	41
9.3. Ext and Bass numbers	42
9.4. v -invariants	44
9.5. Degrees and multiplicities	45
References	46

1. INTRODUCTION

Finiteness conditions on families of algebraic, combinatorial, or geometric objects lead to predictable growth behavior. Typically this growth $n \mapsto P(n)$ is roughly polynomial, or more precisely *quasipolynomial*: there is a subgroup $\Lambda \subseteq \mathbb{Z}$ of finite index and polynomials $\{P_{\overline{m}} \in \mathbb{Q}[n] \mid \overline{m} = m + \Lambda \in \mathbb{Z}/\Lambda\}$ such that

$$P(n) = P_{\overline{m}}(n) \quad \text{for } n \equiv m \pmod{\Lambda}.$$

Example 1.1. Fix a field $\mathbb{k}$. Prototypical polynomial growth arises as follows.

1. Hilbert polynomial: for $R = \bigoplus_{n \in \mathbb{N}} R_n$ noetherian standard-graded over $R_0 = \mathbb{k}$,

$$\dim_{\mathbb{k}}(R_n) = a_d n^d + a_{d-1} n^{d-1} + \cdots + a_0 \in \mathbb{Q}[n] \text{ for } n \gg 0.$$

2. Ehrhart polynomial: for $P \subset \mathbb{Q}^d$ a lattice polytope,

$$\#(nP \cap \mathbb{Z}^d) = b_d n^d + b_{d-1} n^{d-1} + \cdots + b_0 \in \mathbb{Q}[n].$$

3. Snapper polynomial: for any invertible sheaf $\mathcal{L}$ on a dimension d projective scheme X over $\mathbb{k}$, the Euler characteristic χ satisfies

$$\chi(\mathcal{L}^{\otimes n}) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_0 \in \mathbb{Q}[n].$$

These are all honest polynomials, but quasipolynomiality results after

- 1'. removing “standard”, allowing the generators to lie in degrees greater than 1, or
- 2'. changing “lattice” to “rational”, allowing arbitrary integer bounding hyperplanes without requiring the vertices to be integer points.

Further instances in commutative algebra abound for an ideal I in a noetherian ring R :

- 4. the Hilbert–Samuel function $\ell(R/I^n)$ is a polynomial for $n \gg 0$ if I is $\mathfrak{m}$ -primary for a maximal ideal $\mathfrak{m}$;
- 5. the number of generators $\mu(I^n)$ is a polynomial for $n \gg 0$;
- 6. Ext, Tor, Betti and Bass numbers all yield polynomials for $n \gg 0$ [Kod93]:
 - $\ell(\mathrm{Tor}_i^R(R/I^n, M))$,
 - $\ell(\mathrm{Ext}_R^i(M, R/I^n))$,
 - $\mu(\mathrm{Tor}_i^R(R/I^n, M))$,
 - $\mu(\mathrm{Ext}_R^i(M, R/I^n))$;
- 7. the Castelnuovo–Mumford regularity $\mathrm{reg}(R/I^n)$ is linear for $n \gg 0$ when I is homogeneous in a standard-graded ring [Kod00, CHT99];
- 8. the v -invariant $v_{\mathfrak{p}}(I^n) = \min\{|\mathbf{a}| \mid R/I^n \supseteq R/\mathfrak{p} \text{ generated in } \deg \mathbf{a}\}$ is linear for $n \gg 0$ when I is homogeneous in a standard-graded ring [Con24];
- 9. the local cohomology length $\ell(H_{\mathfrak{m}}^i(R/I^n))$ is quasipolynomial for $n \gg 0$ if I is a monomial ideal in a polynomial ring and $\ell(H_{\mathfrak{m}}^i(R/I^n)) < \infty$ for $n \gg 0$ [DM19].

Classically, this quasipolynomial growth derives from the noetherian condition on some related construction, often a Rees algebra or similar. However, quasipolynomial growth has been observed in settings that are demonstrably not noetherian.

Example 1.2. Let $I_n = \langle x^n, y \rangle \subseteq \mathbb{k}[x, y]$ be a graded family of ideals parametrized by $n \in \mathbb{N}$, meaning that $I_n I_m \subseteq I_{n+m}$. The Rees algebra $\bigoplus_{n \in \mathbb{N}} I_n t^n$ is not noetherian. However, the numerical values of classical functors, such as the length

$$\ell(\mathrm{Tor}_1(\mathbb{k}[x, y]/I_n, M))$$

for a fixed noetherian $\mathbb{Z}^2$ -graded $\mathbb{k}[x, y]$ -module M still exhibit quasipolynomial growth. For instance, when $M = \mathbb{k}[x, y]/\langle x^3 y, y^2 \rangle$, the lengths are

$$2, 4, 6, 7, 8, 9, 10, \dots$$

What kind of finiteness drives the conclusion there? Our answer, in the setting of multigraded modules over semigroup rings, melds two themes from recent advances in disparate areas, namely (i) Presburger arithmetic at the intersection of logic, computer science, combinatorics, and polyhedral geometry on one hand, and (ii) tame module theory from applied topology on the other. The upshot is that

- it makes sense for a $\mathbb{Z}^d$ -graded module over a semigroup ring $\mathbb{k}[Q_+]$ with $Q_+ \subseteq \mathbb{Z}^d$ to be constructible in a Presburger sense, materially weaker than noetherian;
- constructible families of such modules exhibit quasipolynomial growth; and
- enough functors preserve constructibility to conclude
- piecewise quasipolynomial growth in surprisingly rich and varied circumstances.

Roughly speaking, after the introduction to Presburger arithmetic in §2, these considerations respectively occupy §4 and 7; §3 and 7.3; §5, 6, and 8; and §7.4, 7.5, and 9. The rest of this Introduction goes into more detail about the methods and background, organized in the same order as the paper is presented.

Presburger arithmetic is the first-order logic of the natural numbers under addition. Presburger introduced it in 1929 [Pre30] and proved its decidability. Its modern importance [Woo15] stems from its ability to describe seemingly complicated sets in the free abelian group $\mathbb{Z}^d$ —for example, defined using combinations of lattices, polyhedra, boolean operations, and quantifiers—in terms of simpler sets, such as translates of affine semigroups with linearly independent generating sets (Theorem 2.10). The consequences for algebraic combinatorics include rationality of generating functions with denominators of the special form familiar to algebraists from Hilbert series [Woo15]. And for enumerative combinatorics, the consequences include ubiquitous quasipolynomial growth, or more precisely piecewise quasipolynomial growth, meaning quasipolynomial behavior on each of finitely many polyhedral regions (Definition 3.8).

The lattice points that Presburger arithmetic organizes into *semisimple sets* (Definition 2.7) serve as multigraded basis vectors for input modules: monomial ideals and their powers, or symbolic powers, or integral closures of powers, and so on. Coarsening the grading to a standard or other linear grading, these modules can be thought of as families of vector spaces whose dimensions grow quasipolynomially because Presburger arithmetic dictates that counting functions behave that way (Theorem 3.11). However, the goal is to conclude quasipolynomial growth for modules output by functors applied to the input modules: Tor, Ext, local cohomology, and so on. These output modules often do not have lattice points as basis elements; instead, the lattice points parametrize a family of finite-dimensional vector spaces that is *constructible* (Definition 4.14) in the sense that the vector spaces vary in Presburger definable ways. More to the point, for enumeration, the family of vector spaces is *numerically constructible* (Definition 3.3) in the sense that the dimensions of the vector spaces are constant on each of finitely many Presburger definable subsets that partition the set of parameter lattice points. Any such numerically constructible family automatically exhibits piecewise quasipolynomial growth (Theorem 3.11).

Numerical constructibility is too weak a condition to hope that it might persist after applying a homological construction, since vector spaces can share dimensions without being naturally isomorphic (see Examples 3.6, 7.16, and 7.17). The key, then, is to identify conditions under which the desired functors preserve not merely numerical constructibility but module-theoretic constructibility of the input. Tensoring with localizations is the easiest and, for our purposes, most elemental example; see §5, which also includes background and notation for combinatorics of subsemigroups of $\mathbb{Z}^d$ and their multigraded modules.

That localization preserves constructibility of modules and morphisms (Lemma 5.12) translates without much fuss into assertions that local cohomology, tensor products,

and higher Tor functors preserve constructibility, as well (Theorems 6.2 and 6.6). By appealing to Matlis duality (Definition 6.8), modules of homomorphisms and higher Ext functors also do (Theorem 6.21).

These functorial preservation results are for individual modules. But the primary interest is in families of modules. Classically these families are indexed by a single integer: a power or symbolic power, for example. One of the benefits of constructible module theory is that it adapts to apply to the parameters of the family as easily as it applies to the grading group of the individual modules in the family. Organizing the members of the family is a *Rees monoid* (Definition 7.1): the Presburger lattice-point analogue of the classical Rees algebra. A constructible family is then simply a constructible module over the Rees monoid algebra (Definition 7.5). Note that the noetherian hypothesis is not relevant; instead the focus is on whether the monoids—and the modules graded by them—are constructible in the (equivalent) semisimple or Presburger senses, for that is what produces piecewise quasipolynomial behavior (Lemma 7.15, given Theorem 3.11).

What results is a host of families of ideals and related objects, easily built from operations like powers, colons, saturation, taking multiplier ideals, integral closures, sums, intersections, or products that are automatically constructible (Theorem 7.21, Proposition 7.22, and Proposition 7.29). Strikingly complicated families emerge from meager building blocks because the results are recursive: they take constructible families for input and guarantee that the output remains constructible. Many of the proofs proceed directly using Presburger arithmetic, avoiding the need to demonstrate directly that various sets are semisimple. An exception is Proposition 7.29, to the effect that the product of a constructible family of ideals with a constructible family of modules is a (doubly indexed) constructible family of modules; its proof instead relies on the general homological theory of modules over posets [Mil25], using the syzygy theorem there [Mil25, Theorem 6.12] to reduce the question to one about Minkowski sums of ideals and arbitrary upsets. These varied techniques offer a sample of the flexibility of constructible module theory, with its complementary roots in logic and applied topology.

The main theoretical results of the paper, which assert functorial preservation of constructibility on families under local cohomology, Tor, and Ext (Theorems 8.1, 8.5, and 8.8) are little more than the join of the corresponding individual module results (§6) and the way families are defined (§7.1–7.2) as modules over Rees monoids. The payoff arrives in §9, with specific piecewise quasipolynomial, quasilinear, or quasiconstant growth statements for length of local cohomology, a -invariants, regularity, and depth (Theorem 9.4); length of Tor and Betti numbers (Theorem 9.10); length of Ext and Bass numbers (Theorem 9.20); associated primes via v -invariants (Theorem 9.27); and extended degrees, including the usual degree, Hilbert–Samuel multiplicity, arithmetic degree, and homological degree (Theorem 9.30, Corollary 9.33, and Remark 9.35).

Comparing our results with prior literature, the general spirit is: most of the quasipolynomial growth results are known for individual functors applied to powers of ideals

or, with a small modification, to noetherian graded families of ideals. Our theory restricts to the multigraded setting, but once there it generalizes by relaxing the noetherian hypothesis on a family of ideals indexed by integers to the much more inclusive constructible hypothesis on a family of modules indexed by integer vectors. Consequently, since the central conceit is that various functors such as Tor , Ext , and local cohomology preserve constructibility (§8, particularly Theorems 8.1, 8.5, and 8.8), any sequence of such functors outputs piecewise quasipolynomial numerical behavior given a constructible input family (§8.4, particularly Corollaries 8.11, 8.12, and 8.13). Rich sources of input constructible families abound (§7.4–7.5) because operations on families of ideals, such as the formation of powers, integral closures, multiplier ideals (Theorem 7.21), colon ideals, saturations, sums, products, and intersections (Proposition 7.22) preserve constructibility.

The algebraic and homological machinery that drives the theory here was conceived for Topological Data Analysis, specifically to subdivide spaces of parameters in persistent homology into subanalytic, semialgebraic, or polyhedral constant regions. Part of the purpose was to connect the multigraded algebra of multiparameter persistence [Mil25, Mil20] to the sheaf-theoretic side [KS18, KS19], which turns out to be more or less equivalent to the subanalytic case [Mil23]. But few hypotheses were needed to enable the homological theory, so it was developed under a unifying “class $\mathfrak{X}$ ” condition analogous to an $\mathcal{o}$ -minimal structure. An integral observation, for the current quasipolynomial purpose, is that the class $\mathfrak{X}$ condition has an arithmetic incarnation: instead of continuously subdividing subsets of real vector spaces into subanalytic or semialgebraic pieces, discretely subdivide subsets of finitely generated free abelian groups into simple pieces.

Certain aspects of semisimple constructibility theory are developed here in their natural generality, namely for a *class* $\mathfrak{X}$ family of subsets (Definition 4.9), because it requires no additional effort. However, the reader interested solely in semisimple (equivalently, by Theorem 2.10, Presburger definable) sets and the resulting semisimple constructibility can always take “class $\mathfrak{X}$ ” to mean “semisimple” or “semisimply constructible”. For this reason, we introduce Presburger arithmetic and semisimple sets (§2) before the general discussion of tameness and class $\mathfrak{X}$ objects and morphisms (§4).

The arithmetic take on subdividing degrees of multigraded modules is not entirely new. It was initiated—in the primitive form of “sector partitions” [HM05]—for algorithmic purposes via monomial matrices [Mil00]. Treating graded degrees of local cohomology as flexible geometric subsets of vector spaces has also seen deep applications to hypergeometric functions [MMW05]. For the present purpose, tracing the idea back to its inception suggests that monomial matrices have the potential to make the current constructible constructions computable, as well, via the general techniques surrounding upset and downset resolutions [Mil25].

Perhaps with algorithmic concerns in mind, a natural next step would be to determine bounds on the constituents of various subdivisions that occur here, including in

semisimple subdivisions numerically subordinate to given modules or families (Definitions 3.1 and 3.2), polyhedral subdivisions that underlie the “piecewise” part of piecewise quasipolynomial (Definition 3.8), and coset partitions that underlie the “quasi” part (Definition 3.7). It would also be valuable to ascertain whether methods based on constructible modules can conclude anything about multigraded regularity. The degrees and leading coefficients of the quasipolynomials in Section 9 vary in ways that would be interesting to study. For example, when the family of functors is indexed by $n \in \mathbb{N}$, constancy in the degree and leading coefficient of one of these quasipolynomials implies existence of the limit of the corresponding sequence over a power of n , which relates to results in the literature [Cut14]. It is also of interest whether any of our results extend to families of modules over graded rings endowed with coarser gradings.

Convention 1.3. Throughout, fix an arbitrary field $\mathbb{k}$. Starting in §5.2, Q denotes a class $\mathfrak{X}$ group (Definition 4.5) unless otherwise stated; see, in particular, the opening of each section. Before that, in §2–§5.1, explicit hypotheses on Q are provided each time. Note that for the family $\mathfrak{X}$ of all subsets of Q , being a class $\mathfrak{X}$ group merely means Q is a full (Definition 4.1) partially ordered subgroup of a finite-dimensional real vector space.

2. PRESBURGER ARITHMETIC

2.1. Presburger definability.

Definition 2.1. A *Presburger formula* is a boolean formula with variables in $\mathbb{Z}$ that can be written using quantifiers ($\exists$ and $\forall$), boolean operations (or, not, and), and integer affine-linear inequalities in the variables.

Henceforth we use bold letters, such as $\mathbf{u}$, to denote either group elements or vectors (of numbers or variables).

Definition 2.2. A variable in a Presburger formula is *free* if it is not quantified. A Presburger formula is written $F(\mathbf{u})$ to indicate that $\mathbf{u}$ is the list of free variables.

Definition 2.3. A set $S \subseteq \mathbb{Z}^d$ is *Presburger definable* if it can be defined via a Presburger formula $F(\mathbf{u})$, meaning that $S = \{\mathbf{u} \in \mathbb{Z}^d \mid F(\mathbf{u})\}$.

Remark 2.4. In any formula defining a Presburger definable set, we sometimes write $F(\mathbf{u}) \Rightarrow G(\mathbf{u})$ to denote $\neg F(\mathbf{u}) \vee G(\mathbf{u})$. The latter notation is justified as the two formulas $F(\mathbf{u}) \Rightarrow G(\mathbf{u})$ and $\neg F(\mathbf{u}) \vee G(\mathbf{u})$ are logically equivalent.

Example 2.5. The positive integers with remainder 1 mod 3 are defined by the formula

$$F(u) = (\exists c \in \mathbb{Z})(c \geq 0 \wedge u = 3c + 1).$$

Note that the variable c in $F(u)$ is not free.

Example 2.6. The formula

$$G(u_1, u_2) = (2u_1 + u_2 \geq 3 \wedge 3u_1 - u_2 \geq 2)$$

defines the set of integer points in the translated cone $(1, 1) + \mathbb{R}_{\geq 0}\Delta$, where Δ is the triangle with vertices $(0, 0)$, $(1, 3)$, $(1, -2)$.

2.2. Decomposing Presburger sets.

Definition 2.7. Given a finitely generated free abelian group Q , a subset $S \subseteq Q$ is *simple* if S is a translate of a subsemigroup of Q isomorphic to $\mathbb{N}^k$ for some k ; that is, $S = \mathbf{q} + N$ for some $\mathbf{q} \in Q$ and N generated by linearly independent elements of Q . The subset $S \subseteq Q$ is *semisimple* if it is a finite disjoint union of simple subsets of Q .

Definition 2.8. An *affine semigroup* is a semigroup that is isomorphic to a finitely generated submonoid of a free abelian group.

Remark 2.9. A semigroup isomorphic to $\mathbb{N}^k$ for some k is often called a *smooth* semigroup (or smooth affine semigroup) because smoothness is equivalent to the semigroup algebra being the coordinate ring of a smooth affine toric variety. A translate of an affine semigroup has been called a *linear* set in the literature, with a finite union of such sets being called *semilinear* [ES69]. The terminology in Definition 2.7 comes from [ES69] and persists in modern developments [DIV12].

The following characterizations of Presburger definable sets summarizes well known results [ES69]; see [DIV12] for a recent viewpoint. Part of a proof is included here because that part is short and may provide insight to readers from commutative algebra.

Theorem 2.10. *For a subset $S \subseteq \mathbb{Z}^d$, the following are equivalent.*

1. S is Presburger definable.
2. S is a finite union of sets of the form $P \cap (\mathbf{q} + Q)$, where $P \subseteq \mathbb{R}^d$ is a rational polyhedron, $\mathbf{q} \in \mathbb{Z}^d$, and Q is a subgroup of $\mathbb{Z}^d$.
3. S is a finite union of translates of affine semigroups.
4. S is semisimple.

In particular, given the final item, any of the unions here can be chosen to be disjoint.

Proof. $1 \Leftrightarrow 2$ by [Woo15, Theorem 1.15].

$2 \Leftrightarrow 3$ by [Sch86, p.234 (19)].

$3 \Leftrightarrow 4$ uses three reductions.

- Every affine semigroup is a disjoint union of translates of normal affine semigroups, where an affine semigroup A is *normal* if $A = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}A$.
- Every rational cone admits a unimodular triangulation: a simplicial subdivision in which each cone is generated by a basis for the ambient lattice.
- Any unimodular triangulation of a cone C induces an expression of the lattice points in C as a finite disjoint union of translates of semigroups each isomorphic to $\mathbb{N}^k$ for some k .

The first reduction follows from [Sta82, Theorem 5.2]; see also [Mil09, Lemma 2.2] for a short proof based on primary decomposition in commutative algebra. The second reduction is a special case of [BG09, Theorem 2.72]. For the third reduction, the translated semigroup associated to each face of the triangulation is the set of lattice points in the relative interior of that face; the relevant semigroup is isomorphic to $\mathbb{N}^k$ because each face is unimodular. $\square$

Remark 2.11. Our proof of Theorem 2.10 can work with an arbitrary—that is, not necessarily unimodular—triangulation: add one simple subset to each lattice point in the fundamental parallelepiped of each simplicial face in the triangulation.

2.3. Semisimple atoms.

Proposition 2.12. *Fix a submonoid A with trivial unit group in a free abelian group Q . Then A is semisimple if and only if the set of atoms of A is semisimple in Q . If A is semisimple and $U \subseteq A$ is an upset (so $\mathbf{a} \in U$ and $\mathbf{a} \preceq \mathbf{a}' \Rightarrow \mathbf{a}' \in U$), then U is semisimple if and only if the minimal generating set of U is semisimple.*

Proof. Fix an isomorphism $Q \xrightarrow{\sim} \mathbb{Z}^d$, and let $\hat{A}$ be the image of A . Assume A is semisimple, so by Theorem 2.10 its image $\hat{A}$ is definable by a Presburger formula $F(\mathbf{u}) = F(u_1, \dots, u_d)$. An atom of $\hat{A}$ is precisely an element that is not the sum of two nonidentity elements of $\hat{A}$. The set of atoms of $\hat{A}$ is thus Presburger via the formula

$$(*) \quad H(\mathbf{u}) = \neg(\exists \mathbf{v}, \mathbf{r} \in \mathbb{Z}^d)(\mathbf{v} \neq \mathbf{0} \wedge \mathbf{r} \neq \mathbf{0} \wedge F(\mathbf{v}) \wedge F(\mathbf{r}) \wedge \mathbf{u} = \mathbf{v} + \mathbf{r}).$$

By Theorem 2.10 again, the set of atoms in A itself is semisimple.

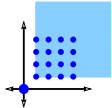
Conversely, let $S \subset \mathbb{Z}^d$ be the set of atoms of $\hat{A}$, and write $S = \bigcup_{i=1}^r (\mathbf{q}_i + N_i)$ with $\mathbf{q}_i \in \mathbb{Z}^d$ and $N_i \subseteq \mathbb{Z}^d$ affine semigroups as in Theorem 2.10. Let F_i be a Presburger formula that defines membership in N_i . Thus $\hat{A}$ is Presburger definable by

$$F(\mathbf{u}) = (\exists n_1, \dots, n_r \in \mathbb{N}, \exists \mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{Z}^d) \left(F_1(\mathbf{a}_1) \wedge \dots \wedge F_r(\mathbf{a}_r) \wedge \mathbf{u} = \sum_{i=1}^r (n_i \mathbf{q}_i + \mathbf{a}_i) \right).$$

By Theorem 2.10 we conclude that $\hat{A}$, and then A , is semisimple.

The proof of the upset claim is similar, using formulas for semisimple subsets of the image $\hat{U}$ of U in $\mathbb{Z}^d$ where needed, such as instead of $F(\mathbf{r})$ in $(*)$ or instead of some of the F_i in the subsequent display. $\square$

Example 2.13. A submonoid $A \subseteq \mathbb{N}^d$ can be infinitely generated but still semisimple.



For instance, $\mathbb{N}^2$ with its axes removed and its origin put back in is not finitely generated but is semisimple.

Example 2.14. Failure to be semisimple arises when the genesis of infinite generation is irrational, such as the submonoid of $\mathbb{N}^2$ on or above the x -axis and below the line $y = \sqrt{2}x$, or nonlinear, such as the submonoid of $\mathbb{N}^2$ on or above the x -axis and below the parabola $y = x^2$.

3. NUMERICS OF CONSTRUCTIBILITY

This section lays out what it means for a family of vector spaces to behave, when counting dimensions, in a semisimple (equivalently, by Theorem 2.10, Presburger definable) manner. The notion of *numerically constructible* family (Definition 3.3) is only a shadow of the module-theoretic constructibility in §7, but it encapsulates the data that extract quasipolynomial behavior (Theorem 3.11) from constructibility.

3.1. Numerical constructibility.

Definition 3.1. A *semisimple subdivision* of a semisimple set S (Definition 2.7 and Theorem 2.10) in a free abelian group Q is an expression of S as a finite disjoint union

$$S = \bigcup_{\alpha \in A} S_\alpha$$

in which each block S_α of the partition is semisimple in Q .

Definition 3.2. Let $V = \bigoplus_{\mathbf{g} \in G} V_{\mathbf{g}}$ be a direct sum of $\mathbb{k}$ -vector spaces indexed by a free abelian group G . A semisimple subdivision of G is *numerically subordinate* to V if $\mathbf{g} \mapsto \dim_{\mathbb{k}} V_{\mathbf{g}}$ is constant on the regions of the subdivision.

Definition 3.3. Let Z be a free abelian group and $\{V_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ be a family of $\mathbb{k}$ -vector spaces each of which is graded by Q , so $V_{\mathbf{n}} = \bigoplus_{\mathbf{q} \in Q} V_{\mathbf{n}\mathbf{q}}$, where $V_{\mathbf{n}\mathbf{q}} = (V_{\mathbf{n}})_{\mathbf{q}}$. The family $\{V_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ over Q is *numerically constructible* if $\dim_{\mathbb{k}} V_{\mathbf{n}\mathbf{q}} < \infty$ for all $\mathbf{n}\mathbf{q} \in G$ and $G = Z \times Q$ admits a semisimple subdivision numerically subordinate to $V = \bigoplus_{\mathbf{n}\mathbf{q} \in G} V_{\mathbf{n}\mathbf{q}}$.

Remark 3.4. To generalize this theory to another class $\mathfrak{X}$ beyond semisimple, Definition 3.3 would need to explicitly require each slice with $\mathbf{n}$ fixed to be class $\mathfrak{X}$. In contrast, the intersection of a semisimple set with $\{\mathbf{n}\} \times Q$ is automatically semisimple.

Remark 3.5. Often the vector spaces $V_{\mathbf{n}}$ are Q -modules that are nonzero only for $\mathbf{n}$ lying in a submonoid of Z that is semisimple as a subset. For example, Z could be $\mathbb{Z}^k$ with the vector spaces $V_{\mathbf{n}}$ only nonzero for $\mathbf{n} \in \mathbb{N}^k$. Graded families of ideals (see Definition 9.14), for instance, have $Z = \mathbb{Z}$ and are only nonzero for $n \in \mathbb{N} \subseteq \mathbb{Z}$.

Example 3.6. Fix $Q = \mathbb{Z}^2$ with positive cone $Q_+ = \mathbb{N}^2$ and $G = \mathbb{Z} \times Q$. Let $I \subset \mathbb{k}[x, y]$ be a monomial ideal and set

$$M_n = \begin{cases} I \oplus \mathbb{k}[x, y]/I & \text{if } n \text{ is even} \\ \mathbb{k}[x, y] & \text{if } n \text{ is odd.} \end{cases}$$

In addition, set $M_n = 0$ for $n < 0$. The family $\{M_n\}_{n \in \mathbb{Z}}$ is numerically constructible because it is subordinate to the subdivision of G with two constant regions:

- the nonnegative octant $\mathbb{N} \times Q_+ \cong \mathbb{N}^3$, where the “Hilbert functions” $\mathbf{g} \mapsto \dim_{\mathbb{k}} V_{\mathbf{g}}$ always take the value 1; and
- elsewhere, where the functions $\mathbf{g} \mapsto \dim_{\mathbb{k}} V_{\mathbf{g}}$ always take the value 0.

3.2. Quasipolynomials from numerical constructibility.

The following definition is standard; see [Woo15, Definitions 1.8 and 1.9], for example.

Definition 3.7. A function $Q : Z \rightarrow \mathbb{Q}$ on a free abelian group Z of rank k is a *quasipolynomial* if there exist a rank k sublattice $\Lambda \subseteq Z$ and polynomials $P_{\bar{\mathbf{m}}} \in \mathbb{Q}[x_1, \dots, x_k]$, one for each coset $\bar{\mathbf{m}} \in Z/\Lambda$, such that $Q(\mathbf{n}) = P_{\bar{\mathbf{m}}}(\mathbf{n})$ for every $\mathbf{n} \in \bar{\mathbf{m}}$.

Definition 3.8. Fix a subset $T \subseteq Z$ in a free abelian group Z . A function $g : T \rightarrow \mathbb{Q}$ is *piecewise quasipolynomial* if there exist

- a finite set A ,
- a partition $Z = \bigcup_{\alpha \in A} (\Gamma_\alpha \cap Z)$ with each Γ_α a rational convex polyhedron, and
- quasipolynomials $Q_\alpha : Z \rightarrow \mathbb{Q}$ for $\alpha \in A$

such that $g(\mathbf{n}) = Q_\alpha(\mathbf{n})$ for every $\mathbf{n} \in \Gamma_\alpha \cap T$. If the polynomials defining the quasipolynomials Q_α are all linear, then Q is *piecewise quasilinear*.

Remark 3.9. In Definition 3.8, if Z has rank 1, then g is piecewise quasipolynomial $\Leftrightarrow g(n)$ coincides with a quasipolynomial for $n \gg 0$ and a quasipolynomial for $n \ll 0$.

Definition 3.10. The *length* of a direct sum of $\mathbb{k}$ -vector spaces $V = \bigoplus_{\mathbf{q} \in Q} V_{\mathbf{q}}$ is $\ell(V) = \sum_{\mathbf{q} \in Q} \dim_{\mathbb{k}} V_{\mathbf{q}}$.

Theorem 3.11. Fix a numerically constructible family $\{V_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ over $Q \cong \mathbb{Z}^d$.

1. The set $T = \{\mathbf{n} \in Z \mid \ell(V_{\mathbf{n}}) < \infty\}$ is semisimple and the function $\ell_V : T \rightarrow \mathbb{N}$ given by $\ell_V(\mathbf{n}) = \ell(V_{\mathbf{n}})$ is piecewise quasipolynomial of degree at most d .
2. For any linear function $\lambda : Q \rightarrow \mathbb{Z}$ the sets

$$\text{top}_\lambda V = \{\mathbf{n} \in Z \mid V_{\mathbf{n}} \neq 0 \text{ and } \sup\{\langle \lambda, \mathbf{q} \rangle \mid V_{\mathbf{nq}} \neq 0\} < \infty\}$$

$$\text{and } \text{soc}_\lambda V = \{\mathbf{n} \in Z \mid V_{\mathbf{n}} \neq 0 \text{ and } \inf\{\langle \lambda, \mathbf{q} \rangle \mid V_{\mathbf{nq}} \neq 0\} > -\infty\}$$

are semisimple, and the functions

$$\begin{aligned} \max_{\lambda, V} : \text{top}_\lambda V &\rightarrow \mathbb{Z} & \text{and } \min_{\lambda, V} : \text{soc}_\lambda V &\rightarrow \mathbb{Z} \\ \mathbf{n} &\mapsto \max_{V_{\mathbf{nq}} \neq 0} \langle \lambda, \mathbf{q} \rangle & \mathbf{n} &\mapsto \min_{V_{\mathbf{nq}} \neq 0} \langle \lambda, \mathbf{q} \rangle \end{aligned}$$

are piecewise quasilinear.

Proof. Let $G = Z \times Q$ and $G = \bigcup_{\alpha \in A} S_\alpha$ be a (finite) semisimple subdivision numerically subordinate to $V = \bigoplus_{\mathbf{nq} \in G} M_{\mathbf{nq}}$. For each $\alpha \in A$ let v_α be the common value of $\dim_{\mathbb{k}} V_g$ for $g \in S_\alpha$. Let $A_0 \subseteq A$ be the set of indices α such that $v_\alpha \neq 0$. For the purpose of writing Presburger formulas in Q , fix an isomorphism $Q \cong \mathbb{Z}^d$.

1. By Theorem 2.10 the sets S_α are defined by Presburger formulas $F_\alpha(\mathbf{n}, \mathbf{q})$. The set T is semisimple because it is defined by the formula (see Remark 2.4 regarding “ $\Rightarrow$ ”)

$$F(\mathbf{n}) = (\exists N \in \mathbb{N}) \left(\bigvee_{\alpha \in A_0} F_\alpha(\mathbf{n}, \mathbf{q}) \Rightarrow \|\mathbf{q}\|_\infty < N \right).$$

For $\mathbf{n} \in T$,

$$\ell_V(\mathbf{n}) = \sum_{\alpha \in A_0} \sum_{\mathbf{q} \in S_\alpha \cap Q_{\mathbf{n}}} \dim_{\mathbb{k}} V_{\mathbf{n}\mathbf{q}} = \sum_{\alpha \in A_0} \# \{ \mathbf{q} \in \mathbb{Z}^d \mid F(\mathbf{n}) \wedge F_\alpha(\mathbf{n}, \mathbf{q}) \} v_\alpha,$$

where $Q_{\mathbf{n}} = \{\mathbf{n}\} \times Q$. For piecewise quasipolynomiality, use [Woo15, Theorem 1.10]. The degree bound follows from the proof of the latter theorem as there are d “counted variables” in our counting functions, namely the coordinates of $\mathbf{q}$ (see also [Stu95]).

2. Fix $\lambda : Q \rightarrow \mathbb{Z}$. The conclusions for $\max_{\lambda, V}$ and $\min_{\lambda, V}$ are similar, so we only write out the case of $\max_{\lambda, V}$. Consider the Presburger formula

$$H(\mathbf{n}, w) = (\exists \mathbf{q} \in \mathbb{Z}^d) \left(\left(\bigvee_{\alpha \in A_0} F_\alpha(\mathbf{n}, \mathbf{q}) \right) \wedge (w = \langle \lambda, \mathbf{q} \rangle) \right).$$

The set $\text{top}_\lambda V$ is defined by the Presburger formula

$$J(\mathbf{n}) = (\exists w \in \mathbb{Z}) H(\mathbf{n}, w) \wedge (\exists N \in \mathbb{N}, \forall w \in \mathbb{Z}) (H(\mathbf{n}, w) \Rightarrow w < N).$$

Consider the “upper” formula

$$U(\mathbf{n}, n) = (\exists w \in \mathbb{Z}) (J(\mathbf{n}) \wedge H(\mathbf{n}, w) \wedge (1 \leq n \leq w))$$

and the “lower” formula

$$L(\mathbf{n}, n) = (\forall w \in \mathbb{Z}) (w > 0 \Rightarrow \neg H(\mathbf{n}, w)) \wedge (\exists w \in \mathbb{Z}) (J(\mathbf{n}) \wedge H(\mathbf{n}, w) \wedge (w \leq n \leq 0)).$$

For $\mathbf{n} \in \text{top}_\lambda V$, if $\max_{V_{\mathbf{n}\mathbf{q}} \neq 0} \langle \lambda, \mathbf{q} \rangle$ is positive, then the expression $\#\{n \mid U(\mathbf{n}, n)\}$ evaluates to $\max_{V_{\mathbf{n}\mathbf{q}} \neq 0} \langle \lambda, \mathbf{q} \rangle$, and it evaluates to zero otherwise. On the other hand, $\#\{n \mid L(\mathbf{n}, n)\}$ evaluates to zero if $\max_{V_{\mathbf{n}\mathbf{q}} \neq 0} \langle \lambda, \mathbf{q} \rangle$ is positive, and it evaluates to $|\max_{V_{\mathbf{n}\mathbf{q}} \neq 0} \langle \lambda, \mathbf{q} \rangle| + 1$ otherwise. Thus, for $\mathbf{n} \in P$,

$$\max_{\lambda, V}(\mathbf{n}) = \#\{n \mid U(\mathbf{n}, n)\} - \#\{n \mid L(\mathbf{n}, n)\} + 1.$$

Hence $\max_{\lambda, V}$ is piecewise quasipolynomial by [Woo15, Theorem 1.10]; the quasipolynomials involved are in fact quasilinear as n is the sole counted variable. $\square$

Remark 3.12. Following the notation in Theorem 3.11, let $S \subset G = Z \times Q$ be a semisimple set. The truncated family $\{V_{\mathbf{n}}|_S\}_{\mathbf{n} \in Z}$, where $V_{\mathbf{n}}|_S = \bigoplus_{\mathbf{n}\mathbf{q} \in S} V_{\mathbf{n}\mathbf{q}}$, is also numerically constructible. The latter is a *linear truncation* of the family $\{V_{\mathbf{n}}\}_{\mathbf{n} \in Z}$. Linearly truncating a given numerically constructible family using a suitable semisimple set can enhance the support of the functions ℓ_V , $\min_{\lambda, V}$, and $\max_{\lambda, V}$ in Theorem 3.11.

4. CONSTRUCTIBLE MODULES

Quasipolynomial behavior arises, in our view, because algebraic objects come in families that are parametrized in a semisimple manner, or more precisely, parametrized by Presburger groups (Definition 4.2). Although our interest is in objects defined by Presburger arithmetic, the proofs are often more transparent when phrased generally in terms of an arbitray “class $\mathfrak{X}$ ” (Definition 4.5), analogous to an $\mathcal{o}$ -minimal structure. These definitions give rise to notions of constructible module (Definitions 4.9 and 4.14).

This section builds foundations that focus on Presburger groups as parametrizing vector spaces. Later, particularly in §7, additional layers of theory allow the Presburger group to be interpreted as parametrizing a family of multigraded modules instead of merely a family of vector spaces.

Definition 4.1. A *partially ordered abelian group* is an abelian group Q with a submonoid Q_+ , the *positive cone*, having trivial unit group. The partial order is given by: $\mathbf{q} \preceq \mathbf{q}' \Leftrightarrow \mathbf{q}' - \mathbf{q} \in Q_+$. If Q_+ generates a subgroup of finite index in Q then Q is *full*.

Definition 4.2. A *Presburger group* is a full partially ordered free abelian group Q of finite rank whose positive cone Q_+ is a semisimple subset of Q .

Example 4.3. If Q_+ is an affine semigroup, then Q is a Presburger group by Theorem 2.10; see also the more general Proposition 2.12.

Remark 4.4. The “full” hypothesis in Definition 4.1 arises in the process of reducing to the case of freely parametrized families (Proposition 7.11).

For many of the statements and proofs, the salient properties of the semisimple hypothesis in Definition 4.2 hold in much greater generality.

Definition 4.5. A full partially ordered abelian group Q is *class* $\mathfrak{X}$ if it is a subgroup of a real vector space of finite dimension and Q_+ belongs to a family $\mathfrak{X}$ of subsets of Q closed under complements, finite unions, negations, and Minkowski sums with Q_+ .

Proposition 4.6. *The semisimple sets form a valid family $\mathfrak{X}$ to define a class $\mathfrak{X}$ group in Definition 4.5.*

Proof. Complements, finite unions, and negatives of Presburger definable sets are Presburger definable. Now, fix an embedding $Q \hookrightarrow \mathbb{Z}^d$. Let $S \subset Q$ be Presburger definable by the formula $F(\mathbf{u})$, and let $G(\mathbf{v})$ be a formula that defines Q_+ . The Minkowski sum $S + Q_+$ is defined by $H(\mathbf{r}) = (\exists \mathbf{u}, \mathbf{v} \in \mathbb{Z}^d)(F(\mathbf{u}) \wedge G(\mathbf{v}) \wedge \mathbf{r} = \mathbf{u} + \mathbf{v})$. Since Presburger definable $\Leftrightarrow$ semisimple by Theorem 2.10, the proof is done. $\square$

Definition 4.7. Fix an arbitrary poset Q . A Q -*module* is a Q -graded $\mathbb{k}$ -vector space $M = \bigoplus_{\mathbf{q} \in Q} M_{\mathbf{q}}$ with a $\mathbb{k}$ -linear map $M_{\mathbf{q}} \rightarrow M_{\mathbf{q}'}$ for every pair $\mathbf{q} \preceq \mathbf{q}'$ in Q such that

- $M_{\mathbf{q}} \rightarrow M_{\mathbf{q}}$ is the identity and
- $M_{\mathbf{q}} \rightarrow M_{\mathbf{q}''}$ is equal to the composite $M_{\mathbf{q}} \rightarrow M_{\mathbf{q}'} \rightarrow M_{\mathbf{q}''}$ if $\mathbf{q} \preceq \mathbf{q}' \preceq \mathbf{q}''$.

Definition 4.7 is equivalent to requiring that M constitute a functor from the partially ordered set Q , viewed as a small category, to the category of vector spaces over $\mathbb{k}$.

Definition 4.8. Let M be a Q -module over a poset Q . A *constant subdivision* of Q subordinate to M is a partition of Q such that for each region I in the partition there exists a vector space M_I and isomorphisms $M_I \rightarrow M_{\mathbf{i}}$ for every $\mathbf{i} \in I$, which has *no monodromy*, i.e., for any pair of regions I, J and any $\mathbf{i} \in I, \mathbf{j} \in J$, if $\mathbf{i} \preceq \mathbf{j}$ then the composition $M_I \rightarrow M_{\mathbf{i}} \rightarrow M_{\mathbf{j}} \rightarrow M_J$ only depend on I and J . In this case we also say that M *dominates* the given partition of Q .

Definition 4.9 ([Mil25, Definition 2.20 and Remark 2.21]). Fix a poset Q .

1. A Q -module M is *tame* if $\dim_{\mathbb{k}} M_{\mathbf{q}}$ is finite for every $\mathbf{q} \in Q$ and Q has a constant subdivision with finitely many regions that is subordinate to M .
2. If Q is a class $\mathfrak{X}$ group, then M is *of class $\mathfrak{X}$* if it is tame via a constant subdivision whose regions lie in the family $\mathfrak{X}$.

Remark 4.10. The notion of tame coincides with class $\mathfrak{X}$ for the family $\mathfrak{X}$ of all subsets of Q , so there is no need to separate claims about tameness from those about class $\mathfrak{X}$.

Definition 4.11. Fix a poset Q . An *upset* $U \subseteq Q$ is a subset closed under going upward in Q (so $\mathbf{q} \in U$ and $\mathbf{q} \preceq \mathbf{q}' \Rightarrow \mathbf{q}' \in U$). Dually, a *downset* $D \subseteq Q$ is a subset closed under going downward in Q . If S is the intersection of an upset and a downset (one of which might be all of Q), then S is *poset-convex*, and $\mathbb{k}\{S\}$ denotes the *indicator module* of S , namely the natural Q -module with basis $\{\mathbf{x}^{\mathbf{q}} \mid \mathbf{q} \in S\}$. When U is an upset, $\mathbb{k}[U]$ is an *upset module*; when D is a downset, $\mathbb{k}[D]$ is a *downset module*.

Remark 4.12. The syzygy theorem for poset modules [Mil25, Theorem 6.12] states many equivalent characterizations of class $\mathfrak{X}$ modules, one of which is that they admit finite presentations and resolutions by class $\mathfrak{X}$ upset or downset modules. The generality of the term “class $\mathfrak{X}$ ” in [Mil25] assumes that the poset Q is a subposet of a “real partially ordered group” [Mil25, Definitions 2.19 and 2.20]. It is an unfortunate oversight that [Mil25, Definition 2.19] assumes the positive cone Q_+ of a real partially ordered group generates Q —unfortunate because that hypothesis is never used in [Mil25]; all that matters is containment in a real vector space (for conditions like “semialgebraic” or “subanalytic”) and existence of an ambient partial order defined by a class $\mathfrak{X}$ positive cone. This is relatively straightforward to verify, because [Mil25] is written for arbitrary posets, so one only needs to search for the phrase “class $\mathfrak{X}$ ” to see that all occurrences (there are less than two dozen) have no use for the “generated by Q_+ ” hypothesis. This slight change in generality is relevant to the “full” hypothesis in Definitions 4.1 and 4.2 because, when Q_+ generates a proper subgroup $\mathbb{Z}Q_+ \subsetneq Q$, the partial order on Q is not induced by any inclusion of Q into a real partially ordered group, as per [Mil25, Definition 2.19], which disallows incomparable distinct cosets of $\mathbb{Z}Q_+$.

Remark 4.13. To say a module is of class $\mathfrak{X}$ means that the module is “constructible”, which for example specializes to semialgebraically constructible or subanalytically constructible when the building blocks are subsets of real vector spaces that are semialgebraic or subanalytic. In the semisimple setting, the building blocks are instead translates of affine semigroups built from a single given lattice $Q \cong \mathbb{Z}^d$ (Definition 2.7).

In analogy with Remark 4.13, the class of semisimple subsets gives rise to a notion of constructible modules.

Definition 4.14. A module M over a Presburger group Q (Definition 4.2) is *semisimply constructible* if it is of class $\mathfrak{X}$ for $\mathfrak{X} = \text{semisimple}$ in Definition 4.9. In this paper, the term *constructible* means semisimply constructible unless otherwise stated.

Remark 4.15. The notion of semisimple here is different than the usual module-theoretic definition of “semisimple” meaning “decomposes as a direct sum of indecomposables”. Arbitrary Q -modules in Definition 4.7 are semisimple in this decomposition sense by the Krull–Remak–Schmidt–Azumaya theorem [BC20, Theorem 1.1].

5. FLAT COMPLEXES OF MODULES

Prior sections set up objects and morphisms with appropriate finiteness conditions (semisimple, Presburger, constructible). This means our *input* modules have appropriately structured (i.e., quasipolynomial, see §8) numerics. What we need now is functors that preserve the finiteness so the *output* modules have similarly structured numerics. This section initiates the process with general results concerning how to construct flat complexes of class $\mathfrak{X}$ (Corollary 5.14) and why such complexes remain of class $\mathfrak{X}$ after tensoring with a module of class $\mathfrak{X}$ (Theorem 5.15).

5.1. Homomorphisms and complexes.

Since the goal is to prove that various functors preserve constructibility, finiteness conditions are needed not merely for objects, but for morphisms and complexes, as well.

Definition 5.1. Fix a poset Q . A *homomorphism* of Q -modules $\varphi : M \rightarrow N$ is a collection of $\mathbb{k}$ -linear maps $M_{\mathbf{q}} \rightarrow N_{\mathbf{q}}$ for $\mathbf{q} \in Q$ making the diagram

$$\begin{array}{ccc} M_{\mathbf{q}} & \longrightarrow & N_{\mathbf{q}'} \\ \downarrow & & \downarrow \\ M_{\mathbf{q}'} & \longrightarrow & N_{\mathbf{q}'} \end{array}$$

commute for every pair $\mathbf{q} \preceq \mathbf{q}'$.

Definition 5.2 ([Mil25, Definition 4.29]). Fix a poset Q . A homomorphism $\varphi : M \rightarrow N$ of Q -modules is *tame* if Q admits a finite constant subdivision subordinate to both M and N such that for each region I in the partition the composite $M_I \rightarrow M_{\mathbf{i}} \rightarrow N_{\mathbf{i}} \rightarrow N_I$ does not depend on $\mathbf{i} \in I$. The constant subdivision is also said to be *subordinate* to φ . If Q and the constant subdivision are class $\mathfrak{X}$ then φ is *of class $\mathfrak{X}$* . If Q is a Presburger group and the constant subdivision is semisimple, then φ is *constructible*.

The notion of upset presentation in the upcoming Definition 5.3 is needed later, for the proof of Proposition 7.29. After that, for the record we state in Lemma 5.5 a criterion to detect when a morphism of modules is of class $\mathfrak{X}$; it patches an oversight in [Mil25, Theorem 6.12], which only stated one direction.

Definition 5.3 ([Mil25, Definition 6.4.1]). Fix a class $\mathfrak{X}$ group Q . An *upset presentation* of a Q -module M is an expression of M as the cokernel of a homomorphism $F_1 \rightarrow F_0$ such that each F_i is a direct sum of finitely many upset modules.

1. An upset presentation is *of class $\mathfrak{X}$* if all of the upsets appearing in it are class $\mathfrak{X}$.

2. A *morphism* of upset presentations is morphism of complexes each of which is an upset presentation.

Remark 5.4. Over an arbitrary poset, it is necessary to require every component of the homomorphism $F_1 \rightarrow F_0$ to be connected in the sense of [Mil25, Definition 3.14]. But partially ordered groups are upper-connected [Mil25, Definition 3.5.2], so all homomorphisms of upset modules are automatically connected [Mil25, Corollary 3.11.2].

Lemma 5.5. *A homomorphism $\varphi : M \rightarrow N$ of Q -modules over a class $\mathfrak{X}$ group Q is of class $\mathfrak{X}$ if and only if it is induced by a morphism of upset presentations.*

Proof. Suppose φ is induced by a morphism of upset presentations. Let $\mathbb{k}[U_1], \dots, \mathbb{k}[U_r]$ be the list of all summands of the four modules involved. Each upset U subdivides Q into two regions, namely U and $Q \setminus U$. The common refinement of these subdivisions is subordinate to φ by the connectedness in Remark 5.4. That proves the “if” direction. The “only if” direction is part of the syzygy theorem [Mil25, Theorem 6.12]. $\square$

Definition 5.6 ([Mil25, Definition 6.17]). Fix a complex $C^\bullet$ of Q -modules.

1. The complex $C^\bullet$ is of class $\mathfrak{X}$ if its modules and morphisms are of class $\mathfrak{X}$.
2. A constant subdivision is *subordinate* to $C^\bullet$ if it is subordinate to all of the modules and morphisms therein, and then $C^\bullet$ is said to *dominate* the subdivision.

5.2. Localization.

An initial premise is that any tensor product in which one of the modules is class $\mathfrak{X}$ and the other is noetherian should result in a class $\mathfrak{X}$ module. However, just a bit more work shows it is enough to demand that one of the modules has a presentation by modules that are finite direct sums of localizations of the monoid algebra $\mathbb{k}[Q_+]$, as long as those localizations are all of class $\mathfrak{X}$. (The noetherian case occurs when $\mathbb{k}[Q_+]$ is noetherian and the localizations are free by virtue of not inverting any nonunit elements.) The next definition introduces this type of module. The relaxation from finitely freely presented to flatly presented is useful because, for example, local cohomology can be computed using complexes of flat modules.

Recall Convention 1.3 concerning the group Q and field $\mathbb{k}$. In particular, Q need not be Presburger unless otherwise stated.

Definition 5.7. Let $S \subseteq Q_+$ be a set that is closed under addition. The *localization* of a Q -module M *along* S is the tensor product

$$M_S = M[\mathbf{x}^{-\mathbf{q}} \mid \mathbf{q} \in S] = M \otimes_{\mathbb{k}[Q_+]} \mathbb{k}[\mathbf{x}^{-\mathbf{q}} \mid \mathbf{q} \in S].$$

Remark 5.8. A Q -module M is equivalently a Q -graded module over the monoid algebra $\mathbb{k}[Q_+]$. As such, the localization $M[m^{-1}]$ of M by inverting an explicit monomial $m = \mathbf{x}^{\mathbf{q}} \in \mathbb{k}[Q_+]$ is the usual notion from commutative algebra.

Definition 5.9. A Q -module M is $\mathfrak{X}$ -flat if it is a finite direct sum of class $\mathfrak{X}$ modules that are translates of localizations of $\mathbb{k}[Q_+]$. The module M is *semisimply flat* if Q is a Presburger group and M is $\mathfrak{X}$ -flat with $\mathfrak{X} = \text{semisimple}$. (The meaning of “semisimply” here refers to Definition 2.7, but in this special context it does not conflict with the direct sum sense in Remark 4.15.)

Definition 5.10. A class $\mathfrak{X}$ set $S \subseteq Q$ is *Minkowski summable* if for any other class $\mathfrak{X}$ set $S' \subseteq Q$ the Minkowski sum $S + S'$ is class $\mathfrak{X}$.

Remark 5.11. For most of the commonly selected classes $\mathfrak{X}$, including semialgebraic, PL, or semisimple (that is, Presburger definable; see Theorem 2.10), every class $\mathfrak{X}$ set is Minkowski summable. The subanalytic class is an exception, although the Minkowski sum of a *bounded* subanalytic set with another subanalytic set is subanalytic. As a concrete class that contains sets that are not Minkowski summable, consider the class $\mathfrak{X}(2)$ consisting of semialgebraic regions defined by polynomials that are quadratic or linear. As long as the positive cone Q_+ is polyhedral, this class $\mathfrak{X}(2)$ is closed under complement, finite intersection, negation, and Minkowski sum with the positive cone Q_+ . However, the Minkowski sum of two quadratic upsets might require quartic (i.e., degree 4) polynomial inequalities.

Lemma 5.12. *Suppose that the positive cone Q_+ is of class $\mathfrak{X}$. Localizing any class $\mathfrak{X}$ module or morphism along a Minkowski summable subset of Q_+ yields a module or morphism of class $\mathfrak{X}$.*

Proof. Localizing $\mathbb{k}\{U\}$ for an upset U along S yields $\mathbb{k}\{U - S\} = \mathbb{k}\{-(-U + S)\}$, which only requires negation and adding S to $-U$. Any class $\mathfrak{X}$ module M admits a class $\mathfrak{X}$ upset presentation by [Mil25, Theorem 6.12] (see Remark 4.12). Localizing any such presentation yields an upset presentation of the localization of M by exactness of localization. If $M \rightarrow N$ is a class $\mathfrak{X}$ morphism, then it lifts to a class $\mathfrak{X}$ upset presentation by the syzygy theorem [Mil25, Theorem 6.12], whose localization along S is an upset presentation of the localized morphism $M_S \rightarrow N_S$, which is class $\mathfrak{X}$ by Definition 5.10 if S is Minkowski summable. $\square$

5.3. $\mathfrak{X}$ -flat complexes.

Proposition 5.13. *Suppose that the positive cone Q_+ is of class $\mathfrak{X}$. If M is of class $\mathfrak{X}$ then any fixed scalar times the localization morphism $M \rightarrow M_S$ along a class $\mathfrak{X}$ Minkowski summable subset S of Q_+ is of class $\mathfrak{X}$.*

Proof. Since S is Minkowski summable, the localized upset $Q_+ - S = -(-Q_+ + S)$ is of class $\mathfrak{X}$. Tensoring an upset presentation of M with the localization morphism $\mathbb{k}[Q_+] \rightarrow \mathbb{k}[Q_+ - S]$ yields a homomorphism of upset presentations which presents the morphism $M \rightarrow M_S$. $\square$

Corollary 5.14. *Any complex in which the modules are $\mathfrak{X}$ -flat is a complex of class $\mathfrak{X}$. Any such complex is called an $\mathfrak{X}$ -flat complex.*

Proof. The content of the statement is that the morphisms are of class $\mathfrak{X}$ automatically if the objects are $\mathfrak{X}$ -flat. By [Mil25, Corollary 3.11.2], all morphisms $\mathbb{k}\{U\} \rightarrow \mathbb{k}\{U'\}$ between indecomposable upset modules are inclusion followed by global scalar multiplication. If $\mathbb{k}\{U\} \rightarrow \mathbb{k}\{U'\}$ is a nonzero morphism of localizations, with $U = Q_+ - S$ and $U' = Q_+ - S'$, then $U \subseteq U'$, by [Mil25, Corollary 3.11.2]. Thus $\mathbb{k}\{U\} \rightarrow \mathbb{k}\{U'\}$ is a natural localization morphism $\mathbb{k}\{U\} \rightarrow \mathbb{k}\{U\}_{S'}$ to which Proposition 5.13 applies. $\square$

Theorem 5.15. *Fix a class $\mathfrak{X}$ group Q , and let $C_\bullet$ be an $\mathfrak{X}$ -flat complex. If M is of class $\mathfrak{X}$, then $C_\bullet \otimes_Q M$ is of class $\mathfrak{X}$, as is the homology of $C_\bullet \otimes M$.*

Proof. By Corollary 5.14, the complex $C_\bullet$ is of class $\mathfrak{X}$. If $\mathbb{k}\{U\} \rightarrow \mathbb{k}\{U'\}$ is a nonzero morphism of indecomposable upset modules, then tensoring this morphism with M yields a scalar multiple of a natural localization map from $\mathbb{k}\{U\} \otimes M \rightarrow \mathbb{k}\{U'\} \otimes M$ by the argument in the proof of Corollary 5.14. Therefore Proposition 5.13 applies. The homology is class $\mathfrak{X}$ by [Mil25, Proposition 4.30.1]. $\square$

5.4. Faces and localizations along them.

Some further notes about localization help clarify the geometry of $\mathfrak{X}$ -flat modules.

Definition 5.16. An *ideal* in a monoid Q_+ is a subset I closed under adding any element of Q_+ , meaning that $\mathbf{q} + I \subseteq I$ for all $\mathbf{q} \in Q$. A *face* of the positive cone Q_+ of a partially ordered abelian group Q is a submonoid $F \subseteq Q_+$ whose complement $Q_+ \setminus F$ is an ideal of the monoid Q_+ . Sometimes it is simpler to say that F is a *face of Q* .

Remark 5.17. The usual definition of prime ideal $\mathfrak{p}$ in a commutative ring R via $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ is equivalent to stipulating that the complement of $\mathfrak{p}$ multiplicative monoid of R is a face: $a \in R \setminus \mathfrak{p}$ and $b \in R \setminus \mathfrak{p} \Rightarrow ab \in R \setminus \mathfrak{p}$.

The notions of localizing by inverting elements and localizing along faces coincide when the positive cone Q_+ is an affine semigroup.

Proposition 5.18. *Assume Q_+ is an affine semigroup. For any Q -module M and monomial $m = \mathbf{x}^{\mathbf{q}} \in \mathbb{k}[Q_+]$, the localization $M[m^{-1}]$ is naturally isomorphic to M_F , where F is the support face of $\mathbf{q}$: the smallest face of Q_+ containing $\mathbf{q}$.*

Proof. There is a natural map $M[m^{-1}] \rightarrow M_F$ whenever $m = \mathbf{x}^{\mathbf{q}}$ with $\mathbf{q} \in F$. If Q_+ is an affine semigroup and $\mathbf{q}$ lies interior to F , then this natural map is an isomorphism because every $\mathbf{f} \in F$ precedes some positive integer multiple of $\mathbf{q}$ in the partial order on Q [MS05, Exercise 7.15]. $\square$

Corollary 5.19. *Fix an affine semigroup Q_+ whose faces are class $\mathfrak{X}$ and Minkowski summable. Assume further that all translates of Q_+ are class $\mathfrak{X}$. Then every $\mathfrak{X}$ -flat module is a finite direct sum of Q -translates of localizations of $\mathbb{k}[Q_+]$ along faces of Q .*

Proof. Immediate from Definition 5.9, Proposition 5.18, and the fact that Q_+ has only finitely many faces. $\square$

Example 5.20. The monoid A depicted in Example 2.13 is “missing” two faces, namely the x - and y -axes. Taking $Q_+ = A$, the Q -modules of any class $\mathfrak{X}$ are closed under localization. Indeed, Q_+ has only two faces: the origin and Q_+ itself. Localizing along the origin is the identity operation. And Q_+ itself is always Minkowski summable by Definition 4.5. There is no way to localize along either of the missing axes.

6. FUNCTORS ON CLASS $\mathfrak{X}$ MODULES

The stage is set to specify hypotheses guaranteeing that various familiar homological functors preserve constructibility (Definitions 4.14 and 5.2), or generally any class $\mathfrak{X}$.

As usual, fix a field $\mathbb{k}$ and class $\mathfrak{X}$ group Q as in Convention 1.3.

6.1. Local cohomology.

Definition 6.1. The *local cohomology* of a Q -module M supported on a monomial ideal $I = \langle m_1, \dots, m_r \rangle \subseteq R = \mathbb{k}[Q_+]$ is the cohomology of $M \otimes \check{C}_I^\bullet$, where

$$\check{C}_I^\bullet : 0 \rightarrow R \rightarrow \bigoplus_{i=1}^n R[m_i^{-1}] \rightarrow \dots \rightarrow \bigoplus_{i_1 < \dots < i_\ell} R[m_{i_1}^{-1} \dots m_{i_\ell}^{-1}] \rightarrow \dots \rightarrow R[m_1^{-1} \dots m_r^{-1}] \rightarrow 0$$

is the Čech complex of $m_1, \dots, m_r$. Write $\check{C}_I^\bullet(M) = M \otimes \check{C}_I^\bullet$.

Theorem 6.2. Fix a monomial ideal $I = \langle m_1, \dots, m_r \rangle$ in the ring $R = \mathbb{k}[Q_+]$, with $m_i = \mathbf{x}^{\mathbf{q}_i}$ for all i . If a Q -module M is of class $\mathfrak{X}$ and, for each i , the semigroup generated by $\mathbf{q}_i$ is Minkowski summable, then the Čech complex $\check{C}_I^\bullet(M)$ is of class $\mathfrak{X}$, as is the cohomology $H_i^\bullet(M)$ of $\check{C}_I^\bullet(M)$.

Proof. The localizations in $\check{C}_I^\bullet$ are of class $\mathfrak{X}$ by Lemma 5.12. Hence $\check{C}_I^\bullet$ is of class $\mathfrak{X}$ by Proposition 5.13. The theorem is therefore a special case of Theorem 5.15. $\square$

Remark 6.3. The Minkowski summable hypothesis of Theorem 6.2 is always satisfied when $\mathfrak{X} = \text{semialgebraic}$, PL , or semisimple , or when $\mathfrak{X}$ is the family of all subsets of Q .

6.2. Tensor products and Tor.

Example 6.4. Localizing along a Minkowski summable set is a particular case of tensor product. General tensor products do not preserve class $\mathfrak{X}$ or even tameness of any sort. An example is the indicator $\mathbb{Z}^2$ -module $\mathbb{k}\{L\}$ supported on the antidiagonal line $L = \{(x, y) \mid y = -x\}$. The tensor product of this module with itself over $\mathbb{k}[\mathbb{N}^2]$ is

$$\mathbb{k}\{L\} \otimes \mathbb{k}\{L\} \cong \bigoplus_{\mathbf{a} \in L} \mathbb{k}\{L\},$$

so for each $\mathbf{a} \in L$, the vector space $(\mathbb{k}\{L\} \otimes \mathbb{k}\{L\})_{\mathbf{a}}$ in degree $\mathbf{a} = (a, -a)$ is the direct sum $\bigoplus_{i \in \mathbb{Z}} \mathbb{k}\{\mathbf{a} + \mathbf{i}\} \otimes \mathbb{k}\{-\mathbf{i}\}$ of tensor products of the 1-dimensional vector spaces at $\mathbf{a} + \mathbf{i} = (a + i, -a - i)$ and $-\mathbf{i} = (-i, i)$.

This phenomenon recurs along the antidiagonal L in the tensor product $\mathbb{k}\{U\} \otimes \mathbb{k}\{U\}$, where $U = L + \mathbb{N}^2$ is the upset generated by L . Thus tensor products of individual upset modules with individual upset modules need not be tame, let alone class $\mathfrak{X}$ for any particular $\mathfrak{X}$. Note that these examples of $\mathbb{k}\{L\}$ and $\mathbb{k}\{U\}$ are constructible modules.

Example 6.4 notwithstanding, tensor products or Tor in which one of the input modules is class $\mathfrak{X}$ and the other is noetherian still result in class $\mathfrak{X}$ output. In fact, more is true: it is enough to demand that one of the modules has class $\mathfrak{X}$ flat resolution.

Definition 6.5. Henceforth $\otimes_Q$ and Tor_i^Q denote $\otimes_{\mathbb{k}[Q_+]}$ and $\text{Tor}_i^{\mathbb{k}[Q_+]}$, respectively.

Theorem 6.6. Fix a class $\mathfrak{X}$ group Q . If M is any Q -module of class $\mathfrak{X}$ and N is a Q -module admitting a flat resolution $C_{\bullet}$ that is $\mathfrak{X}$ -flat in homological degrees $i - 1$, i , and $i + 1$, then $\text{Tor}_i^Q(M, N)$ is of class $\mathfrak{X}$.

Proof. $\text{Tor}_i^Q(M, N)$ can be computed as the middle homology of the three-term complex $M \otimes_Q C_{i-1} \leftarrow M \otimes_Q C_i \leftarrow M \otimes_Q C_{i+1}$, which is of class $\mathfrak{X}$ by Theorem 5.15. $\square$

Remark 6.7. Assume Q_+ affine semigroup and that all translates of Q_+ are class $\mathfrak{X}$. In Theorem 6.6, if N is noetherian, then any free resolution with finite rank in each homological degree is $\mathfrak{X}$ -flat in all homological degrees. Assume further that every face of Q_+ is class $\mathfrak{X}$ and Minkowski summable. If N is artinian, then its Matlis dual $N^\vee$ (Definition 6.8) is noetherian, and so by [MS05, Theorem 11.30] $N^\vee$ has an injective resolution whose modules are Matlis duals of $\mathfrak{X}$ -flat modules (see also Remark 6.18). Taking Matlis dual of such resolution and using Lemma 6.13 proves that any artinian N admits a flat resolution $C_{\bullet}$ that is $\mathfrak{X}$ -flat in all homological degrees. Note that all these assumptions on Q_+ are satisfied when $\mathfrak{X} = \text{semisimple}$.

6.3. Hom and Ext.

The results concerning flat complexes and Tor perhaps surprisingly can be massaged to work for Ext, as well, despite Ext being asymmetric in its two arguments. The main tool is Matlis duality, which behaves extremely well when the relevant vector space dimensions are all finite, as is the case when constructibility (or, more generally, class $\mathfrak{X}$) is involved. The following review of Matlis duality is based on [MS05, §11.3].

Definition 6.8. The *Matlis dual* of a Q -module M is the Q -module $M^\vee$ defined by

$$(M^\vee)_{\mathbf{q}} = \text{Hom}_{\mathbb{k}}(M_{-\mathbf{q}}, \mathbb{k}),$$

so the homomorphism $(M^\vee)_{\mathbf{q}} \rightarrow (M^\vee)_{\mathbf{q}'}$ is transpose to $M_{-\mathbf{q}'} \rightarrow M_{-\mathbf{q}}$.

The degree-by-degree formula for $M^\vee$ becomes more transparent when all degrees are considered simultaneously via the following standard notion of graded Hom.

Definition 6.9. Fix a Q -graded ring R , such as $\mathbb{k}$ (concentrated in degree $\mathbf{0}$) or $\mathbb{k}[Q_+]$. For any Q -graded R -modules M and N , set

$$\underline{\mathrm{Hom}}_R(M, N)_{\mathbf{q}} = \text{degree } \mathbf{q} \text{ homogeneous } R\text{-homomorphisms } M \rightarrow N$$

in which $M_{\mathbf{a}} \rightarrow N_{\mathbf{a}+\mathbf{q}}$ for all $\mathbf{a}$. The *graded Hom* is

$$\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_{\mathbf{q} \in Q} \underline{\mathrm{Hom}}_R(M, N)_{\mathbf{q}}$$

defined as a Q -graded $\mathbb{k}[Q_+]$ -module by $(\mathbf{x}^{\mathbf{q}}\varphi)(m) = \varphi(\mathbf{x}^{\mathbf{q}}m)$ for $\varphi \in \underline{\mathrm{Hom}}_R(M, N)$.

Definition 6.10. Write Hom_Q and Ext_Q^i to mean $\mathrm{Hom}_{\mathbb{k}[Q_+]}$ and $\mathrm{Ext}_{\mathbb{k}[Q_+]}^i$, respectively.

Proposition 6.11. *If $F_{\bullet}$ is any complex of Q -modules with Matlis dual $I^{\bullet} = F_{\bullet}^{\vee}$, and $C_{\bullet}$ is any complex of Q -modules, then canonically*

$$\underline{\mathrm{Hom}}_Q(C_{\bullet}, I^{\bullet}) \cong (C_{\bullet} \otimes_Q F_{\bullet})^{\vee}.$$

Proof. This is [MS05, Lemma 11.16], which is stated for modules but works as well for complexes by functoriality of $\underline{\mathrm{Hom}}$ and $\otimes$. Also note that the proof there—and hence the statement here—is a simple abstract adjunction with no noetherian requirement. $\square$

Corollary 6.12. *For any Q -modules M and N , there is a natural isomorphism*

$$\underline{\mathrm{Ext}}_Q^i(M, N^{\vee}) = \mathrm{Tor}_i^Q(M, N)^{\vee}.$$

Proof. Apply Proposition 6.11 to flat resolutions $C_{\bullet}$ and $F_{\bullet}$ of M and N , respectively, so the Matlis dual $I^{\bullet} = (F_{\bullet})^{\vee}$ is an injective resolution of $N^{\vee}$. Actually, it suffices to let $F_{\bullet}$ be a flat resolution of N :

$$\begin{aligned} \underline{\mathrm{Ext}}_Q^i(M, N^{\vee}) &= H^i \underline{\mathrm{Hom}}_Q(M, I^{\bullet}) \\ &= H^i \underline{\mathrm{Hom}}_Q(M, \underline{\mathrm{Hom}}_{\mathbb{k}}(F_{\bullet}, \mathbb{k})) \\ &= H^i \underline{\mathrm{Hom}}_{\mathbb{k}}(M \otimes_Q F_{\bullet}, \mathbb{k}) \\ &= \underline{\mathrm{Hom}}_{\mathbb{k}}(H_i(M \otimes_Q F_{\bullet}), \mathbb{k}) \\ &= H_i(M \otimes_Q F_{\bullet})^{\vee} \\ &= \mathrm{Tor}_i^Q(M, N)^{\vee}. \end{aligned} \quad \square$$

Lemma 6.13. *Matlis duality preserves class $\mathfrak{X}$: if M is of class $\mathfrak{X}$ then so is $M^{\vee}$. Moreover, in that case, $(M^{\vee})^{\vee} \cong M$, as in any case where $\dim_{\mathbb{k}} M_{\mathbf{q}} < \infty$ for all $\mathbf{q} \in Q$.*

Proof. If a subdivision is subordinate to M , then taking the negative (in Q) of each region in the subdivision yields a subdivision subordinate to $M^{\vee}$ that is class $\mathfrak{X}$ by the negation clause in Definition 4.9. The claim about $(M^{\vee})^{\vee}$ follows from the fact that any vector space of finite dimension is naturally isomorphic to its double dual; note that class $\mathfrak{X}$ implies tame (Definition 4.9), which includes this finite dimensionality. $\square$

Remark 6.14. Matlis duality is exact (it is vector space duality, degree by degree) and it swaps flat and injective modules [MS05, Lemma 11.23]. Note that $\mathbb{k}[Q_+]$ does not need to be noetherian for the brief adjunction argument there.

The following lemma gives a structural result about tame injective modules, or more generally (Remark 4.10) injective modules of class $\mathfrak{X}$.

Lemma 6.15. *Every class $\mathfrak{X}$ injective Q -module I is a finite direct sum of class $\mathfrak{X}$ indecomposable injective modules.*

Proof. Since I is tame by Definition 4.9, $\dim_{\mathbb{k}} M_{\mathbf{q}}$ is finite for every $\mathbf{q} \in Q$ also by Definition 4.9. Therefore, the Krull–Remak–Schmidt–Azumaya theorem [BC20, Theorem 1.1] yields a decomposition $I = \bigoplus_{\lambda \in \Lambda} I_{\lambda}$ with each I_{λ} an indecomposable Q -module. Being a direct summand of an injective module, each I_{λ} is also injective. For the first claim, it remains is to show that Λ is finite.

Fix I_{λ} for some $\lambda \in \Lambda$ and a non-zero element $a \in (I_{\lambda})_{\mathbf{q}_0}$ for some $\mathbf{q}_0 \in Q$. Multiplication by $\mathbf{x}^{\mathbf{q}_0}$ induces a degree $\mathbf{0}$ homomorphism $\mathbb{k}\{\mathbf{q}_0 + Q_+\} \rightarrow I_{\lambda}$ that extends to a degree $\mathbf{0}$ homomorphism $\varphi : \mathbb{k}\{Q\} \rightarrow I_{\lambda}$ because I_{λ} is injective. Since the image of φ is a quotient of $\mathbb{k}\{Q\}$, it is a downset module $\mathbb{k}\{D_{\lambda}\} \cong \text{im}(\varphi) \subseteq I_{\lambda}$. But every finite collection of downsets in Q has nonempty intersection (this is elementary to prove directly; partially ordered abelian groups are “lower-connected” in the language of [Mil25, Definition 3.5.3]) and $\dim_{\mathbb{k}} M_{\mathbf{q}}$ is bounded thanks to any subordinate constant subdivision which has only finitely many constant regions. It follows that Λ is finite.

The second claim is by Lemma 6.16, which is separated off for the record. $\square$

Lemma 6.16. *Assume that a Q -module $M \cong \bigoplus_{\lambda \in \Lambda} M_{\lambda}$ is decomposable as a direct sum. If M is tame or of class $\mathfrak{X}$, then so is M_{λ} for each index λ .*

Proof. The direct sum induces projection maps $M_{\mathbf{a}} \rightarrow (M_{\lambda})_{\mathbf{a}}$ for all $\mathbf{a} \in Q$. The structure maps of M (multiplications by monomials) induce structure maps on each of its summands. Any constant subdivision of Q subordinate to M is therefore automatically subordinate to M_{λ} , by direct check of Definition 4.8. $\square$

Definition 6.17. A Q -module M is $\mathfrak{X}$ -injective if its Matlis dual $M^{\vee}$ (Definition 6.8) is $\mathfrak{X}$ -flat (Definition 5.9). An $\mathfrak{X}$ -injective Q -module is *semisimply injective* if Q is a Presburger group and $\mathfrak{X} = \text{semisimple}$. (Note again that the meaning of “semisimply” here refers to Definition 2.7; cf. Definition 5.9.)

Remark 6.18. If Q_+ is an affine semigroup, then by [MS05, Theorem 11.30] the injective $\mathbb{k}[Q_+]$ -modules are direct sums of (possibly infinitely many) indecomposable injectives of the form $\mathbb{k}\{\mathbf{q} + F - Q_+\}$ for some $\mathbf{q} \in Q$ and face F of Q_+ (Definition 5.16). By Lemma 6.15 the number of indecomposables is finite if the injective module is tame.

Example 6.19. Over a general class $\mathfrak{X}$ partially ordered abelian group Q , there could be a distinction between $\mathfrak{X}$ -injective Q -modules and injective Q -modules of class $\mathfrak{X}$, because a class $\mathfrak{X}$ indecomposable injective need not be Matlis dual to a localization along a face. Take $Q = \mathbb{R}^n$, for instance, with $Q_+ = (\mathbb{R}_{\geq 0})^n$. If $U = (\mathbb{R}_{> 0})^n$ is the interior of the positive orthant, then the upset module $\mathbb{k}[U]$ is flat [GM23, Definition 2.2 and Proposition 2.4] and indecomposable but not a localization of $\mathbb{k}[Q_+]$.

Remark 6.20. Despite the caveat in Example 6.19, using only $\mathfrak{X}$ -injectives instead of arbitrary injectives of class $\mathfrak{X}$ is not restrictive in the cases of interest here, because discrete settings like Presburger definable behave more like Remark 6.18 than Example 6.19. Indeed, using Lazard’s criterion [Laz64] to express flat modules as filtered colimits of free modules, discrete multigradings force degrees to head off to infinity. The failure of discreteness in Example 6.19 is decisive, and it can cause far worse behavior than oddly shaped flat or injective modules: tame modules need not admit flat presentations with countably many summands [Mil20, Example 1.2], let alone finitely many localization summands. Tame modules do, however, always admit finite resolutions by finite direct sums of upset or downset modules [Mil25, Theorem 6.12], so numerics in these more general situations are not entirely hopeless.

Theorem 6.21. *Fix a class $\mathfrak{X}$ group Q . If one of the following two scenarios is in effect for given Q -modules M and N , then $\underline{\mathrm{Ext}}_Q^i(M, N)$ is of class $\mathfrak{X}$.*

1. *M is of class $\mathfrak{X}$ and N has an injective resolution $I^\bullet$ that is $\mathfrak{X}$ -injective in cohomological degrees $i - 1$, i , and $i + 1$.*
2. *N is of class $\mathfrak{X}$ and M has a flat resolution $C_\bullet$ that is $\mathfrak{X}$ -flat in homological degrees $i - 1$, i , and $i + 1$.*

Proof. By Remark 6.14, in Scenario 1, $F_\bullet = (I^\bullet)^\vee$ is a flat resolution of $N^\vee$. The $\mathfrak{X}$ -injective hypothesis in cohomological degrees surrounding i imply that the three-term complex $I^{i-1} \rightarrow I^i \rightarrow I^{i+1}$ is Matlis dual to $F_{i-1} \leftarrow F_i \leftarrow F_{i+1}$ by the double-dual part of Lemma 6.13. Thanks to Proposition 6.11, $\underline{\mathrm{Ext}}_Q^i(M, N)$ is Matlis dual to the middle homology of the three-term complex $M \otimes_Q F_{i-1} \leftarrow M \otimes_Q F_i \leftarrow M \otimes_Q F_{i+1}$. The resulting cohomology is of class $\mathfrak{X}$ by Theorem 5.15 and Lemma 6.13.

In Scenario 2, $\underline{\mathrm{Ext}}_Q^i(M, N)$ can be computed as the Matlis dual of the middle homology of the three-term complex $C_{i-1} \otimes_Q N^\vee \rightarrow C_i \otimes_Q N^\vee \rightarrow C_{i+1} \otimes_Q N^\vee$ by Proposition 6.11 and the double-dual part of Lemma 6.13 (the latter applied to N). The resulting cohomology is of class $\mathfrak{X}$ by Theorem 5.15 and Lemma 6.13. $\square$

Remark 6.22. Assume that Q_+ is an affine semigroup whose faces are class $\mathfrak{X}$ and Minkowski summable. Assume further that all translates of Q_+ are class $\mathfrak{X}$. (These assumptions on Q_+ are all satisfied when $\mathfrak{X} = \text{semisimple}$.)

1. The assumption on $I^\bullet$ in Theorem 6.21.1 is satisfied in all cohomological degrees by any noetherian or artinian N by Remarks 6.14, 6.7, and 6.18.

2. The assumption on $C_\bullet$ in Theorem 6.21.2 is satisfied in all homological degrees by any noetherian or artinian M as shown in Remark 6.7.

7. CONSTRUCTIBLE FAMILIES

This section formulates constructions to extend the notions of (semisimple) constructibility from modules to families of modules. It essentially takes the theoretical join of §3 and 4, which respectively treat families at a numerical level and constructibility for individual modules as opposed to in families.

In this section, Q is a Presburger group (Definition 4.2) unless otherwise specified.

7.1. Rees monoids.

Classically, the Rees algebra of a family of ideals lends control over numerics, if the Rees algebra is noetherian. Here, what lends that control is the Rees monoid.

Definition 7.1. A *Rees monoid* over Q is the positive cone G_+ of a Presburger group $G = Z \times Q$ such that $G_+ \cap (\{0\} \times Q) = \{0\} \times Q_+$ (so the zero-slice of the positive cone G_+ is a copy of Q_+). A Rees monoid G_+ over Q is *free* or *flat* if, respectively, the monoid algebra $\mathbb{k}[G_+]$ is free or flat as a module over $\mathbb{k}[Q_+]$.

The rubric for Definition 7.1 is the quintessential construction of the Rees algebra from powers of an ideal. It is worth isolating this special case. It is also useful to widen the context to allow independent powers of finitely many ideals.

Definition 7.2. The *Rees algebra* of a set of monomial ideals $J_1, \dots, J_k$ in $\mathbb{k}[Q_+]$ is the algebra $\mathbb{k}[Q_+][J_1 t_1, \dots, J_k t_k] \subseteq \mathbb{k}[Q_+][t_1, \dots, t_k]$, where $t_1, \dots, t_k$ are indeterminates.

Example 7.3. Fix nonzero constructible monomial ideals $J_1, \dots, J_k$ of $\mathbb{k}[Q_+]$. If $G_+ \subseteq \mathbb{N}^k \times Q_+$ is the semigroup whose monoid algebra $\mathbb{k}[G_+]$ equals $\mathbb{k}[Q_+][J_1 t_1, \dots, J_k t_k]$, then by Proposition 2.12 G_+ is a Rees monoid over Q , with $G = \mathbb{Z}G_+ = \mathbb{Z}^k \times Q$: the atoms of G_+ correspond to atoms of Q_+ and minimal generators of $J_1, \dots, J_k$.

Definition 7.4. In Example 7.3, G_+ is the *Rees monoid* of the ideals $J_1, \dots, J_k$.

7.2. Constructible families.

Classically, a family yields a module over the Rees algebra. The same idea works here. Added freedom in this multigraded context arises because this Rees algebra module need not be noetherian; it only needs to be definable by Presburger arithmetic.

Definition 7.5. Fix a Rees monoid G_+ over Q with $G = Z \times Q$. A family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ of Q -modules is a *G-family* if its direct sum yields a G -module

$$\mathcal{M} = \bigoplus_{\mathbf{n} \in Z} M_{\mathbf{n}}(-\mathbf{n}) = \bigoplus_{\mathbf{g} \in G} M_{\mathbf{g}} = \bigoplus_{\mathbf{n} \mathbf{q}} M_{\mathbf{n} \mathbf{q}}$$

where $M(-\mathbf{n})$ is the Q -module M shifted to the slice $\{\mathbf{n}\} \times Q$ of G . A *constructible family* over G_+ is a G -family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ whose associated G -module $\mathcal{M}$ is constructible.

Remark 7.6. In Definition 7.5, if Z has rank 0 then a G -family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ consists of a single Q -module M , which is constructible when the family is constructible.

The simplest examples of constructible families are those over Rees algebras of finitely many constructible monomial ideals. It is useful to develop methods of detecting this constructibility.

Definition 7.7. Fix a family $\mathbb{M} = \{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ of indicator Q -modules indexed by a free abelian group Z , so $M_{\mathbf{n}} = \mathbb{k}\{S_{\mathbf{n}}\}$ for some poset-convex subset $S_{\mathbf{n}} \subseteq Q$ (Definition 4.11). To say *membership in $\mathbb{M}$ is Presburger* means that the set $\bigcup_{\mathbf{n} \in Z} \{\mathbf{n}\} \times S_{\mathbf{n}} = \{\mathbf{nq} \in Z \times Q \mid \dim_{\mathbb{k}} M_{\mathbf{nq}} = 1\}$ is semisimple.

Lemma 7.8. *Membership in a family $\mathbb{I} = \{I_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ of monomial ideals is Presburger if and only if the top set $T \subset Z \times Q_+$ of all $\mathbf{nq}$ such that $\mathbf{x}^{\mathbf{q}}$ is a minimal monomial generator of $I_{\mathbf{n}}$ is semisimple.*

Proof. Choose isomorphisms $Q \cong \mathbb{Z}^d$ and $Z \cong \mathbb{Z}^k$, and use Theorem 2.10 to pass freely between “semisimple” and “Presburger definable”. Let Q_+ be defined by the Presburger formula $G(\mathbf{v})$. Assume membership in $\mathbb{I}$ is definable by $F(\mathbf{n}, \mathbf{q})$. The top set T can be defined via the formula

$$H(\mathbf{n}, \mathbf{q}) = \neg(\exists \mathbf{a}, \mathbf{v} \in \mathbb{Z}^d)(F(\mathbf{n}, \mathbf{a}) \wedge G(\mathbf{v}) \wedge \neg(\mathbf{q} = \mathbf{0}) \wedge (\mathbf{q} = \mathbf{a} + \mathbf{v})).$$

Conversely, assume T is defined by $H(\mathbf{n}, \mathbf{a})$. Membership in $\mathbb{I}$ is Presburger definable by the formula

$$F(\mathbf{n}, \mathbf{q}) = (\exists \mathbf{a}, \mathbf{v} \in \mathbb{Z}^d)(H(\mathbf{n}, \mathbf{a}) \wedge G(\mathbf{v}) \wedge (\mathbf{q} = \mathbf{a} + \mathbf{v})). \quad \square$$

Proposition 7.9. *Fix a family $\mathbb{M} = \{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ of indicator Q -modules and nonzero constructible monomial ideals $J_1, \dots, J_k$ of $\mathbb{k}[Q_+]$. The following are equivalent.*

1. $\mathbb{M}$ is constructible over the Rees monoid of $J_1, \dots, J_k$.
2. Membership in $\mathbb{M}$ is Presburger and $J_i \subseteq \bigcap_{\mathbf{n} \in \mathbb{Z}^k} M_{\mathbf{n} + \mathbf{e}_i} :_R M_{\mathbf{n}}$ for every i , where $\mathbf{e}_i$ is the i th standard basis vector of $\mathbb{Z}^k$.

Proof. The Rees monoid G_+ of $J_1, \dots, J_k$ makes $G = \mathbb{Z}G_+$ a Presburger group because the ideals are constructible. This allows the notion of constructible G -family to make sense in the first place.

Now define $\mathcal{M} = \bigoplus_{\mathbf{n} \in \mathbb{Z}^k} M_{\mathbf{n}}(-\mathbf{n})$. The condition for $\mathcal{M}$ to be a G -module is that $J_i M_{\mathbf{n}} \subseteq M_{\mathbf{n} + \mathbf{e}_i}$ for every i and all $\mathbf{n} \in \mathbb{Z}^k$. This condition is equivalent to requiring that $J_i \subseteq \bigcap_{\mathbf{n} \in \mathbb{Z}^k} M_{\mathbf{n} + \mathbf{e}_i} :_R M_{\mathbf{n}}$ for every i . Moreover, $\mathcal{M}$ dominates a constant subdivision of G with two regions $S_0 = \{\mathbf{g} \in G \mid \mathcal{M}_{\mathbf{g}} = 0\}$ and $S_1 = \{\mathbf{g} \in G \mid \dim_{\mathbb{k}} \mathcal{M}_{\mathbf{g}} = 1\}$ because the supports of the Q -slices $M_{\mathbf{n}}$ are poset-convex by Definition 4.11. (The poset-convexity ensures that structure homomorphisms $\mathbb{k} = \mathcal{M}_{\mathbf{g}} \rightarrow \mathcal{M}_{\mathbf{g}'} = \mathbb{k}$ are all isomorphisms and not forced to factor through some $\mathcal{M}_{\mathbf{g}''} = 0$.) This subdivision is semisimple precisely when membership in $\mathbb{M}$ is Presburger definable. $\square$

Corollary 7.10. *If the family $\mathbb{M}$ is constructible over the Rees monoid of constructible monomial ideals $J_1, \dots, J_k$, and $J'_i \subseteq J_i$ are nonzero constructible monomial ideals, then $\mathbb{M}$ is also constructible over the Rees monoid of $J'_1, \dots, J'_k$.*

Proof. Proposition 7.9.2 immediately passes to the sub-ideals J'_i . $\square$

The following proposition allows us to restrict to free (and hence flat) Rees monoids when dealing with tensor products in the proof of Theorem 8.5.

Proposition 7.11. *Each constructible family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ over a Rees monoid G_+ over Q , with $G = Z \times Q$, has a free reduction: a free Rees monoid $G'_+ \subseteq G_+$ over Q , with $G' = Z' \times Q$ of the same rank as G , such that $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ is a constructible family over G' .*

Proof. Let k be the rank of Z and $\pi : G = Z \times Q \rightarrow Z$ be projection. Since G_+ generates a finite index subgroup of G by the full hypothesis in Definition 7.1 (see Definition 4.1), there exist k linearly independent elements $\mathbf{n}_1, \dots, \mathbf{n}_k$ in $\pi(G_+)$. Denoting each ordered pair $(\mathbf{n}, \mathbf{q}) \in Z \times Q$ by $\mathbf{nq}$, for notational simplicity (and to match the subscripts in Definition 7.5), for each i choose $\mathbf{n}_i \mathbf{q}_i \in G_+ \cap \pi^{-1}(\mathbf{n}_i)$. Let $G'_+ \subseteq G_+$ be the monoid generated by $\{\mathbf{0}\} \times Q$ and the $\mathbf{n}_i \mathbf{q}_i$. By construction, G'_+ is a free Rees monoid over Q . In particular, G'_+ is semisimple, and $G' = \mathbb{Z}G'_+ = Z' \times Q$ has the same rank as G , where $Z' = \mathbb{Z}\{\mathbf{n}_1, \dots, \mathbf{n}_k\}$.

The restriction $M_{\mathbf{z}+Z'}$ of $\mathcal{M}$ to the coset of Z' based at $\mathbf{z} \in Z$ is a G' -family because $\mathcal{M}$ is a G -family and $G'_+ \subseteq G_+$. Moreover, if $G = \bigcup_{\alpha \in A} S_{\alpha}$ is a semisimple constant subdivision of G subordinate to $\mathcal{M}$, then intersecting each S_{α} with the coset $\mathbf{z} + Z'$ yields a semisimple constant subdivision of G subordinate to $M_{\mathbf{z}+Z'}$ after appending one additional constant region $Z \setminus (\mathbf{z} + Z')$ where $M_{\mathbf{z}+Z'}$ vanishes. Hence $M_{\mathbf{z}+Z'}$ is constructible, and so is $\mathcal{M}$, being the finite direct sum of these over the cosets of Z' . $\square$

Example 7.12. View $Q_+ = \mathbb{N}^2$ as lying in a horizontal plane, and let G_+ be the cone over a “square”, namely generated by Q_+ and the two vectors $(1, 0, 1)$ and $(0, 1, 1)$ directly above the two generators of Q_+ . The submonoid G'_+ in Proposition 7.11 could be generated by, say, Q_+ and the vector $(1, 0, 1)$ that sits above the x -axis. Although any constructible family over G_+ is automatically constructible over G'_+ by the Proposition, it is not true that a noetherian G -module must be noetherian as a G' -module. Indeed, even $\mathbb{k}[G_+]$ itself fails to be finitely generated over $\mathbb{k}[G'_+]$. Constructibility is a weaker—meaning more inclusive—absolute combinatorial condition, whereas noetherian is a stronger relative algebraic property that depends strongly on the base ring.

One strength of constructible module theory is the ease of extending it to complexes.

Definition 7.13. Fix a Rees monoid G_+ over Q with $G = Z \times Q$. A *constructible family of complexes* is a family $\{C_{\mathbf{n}}^{\bullet}\}_{\mathbf{n} \in Z}$ of complexes of Q -modules whose direct sum $C^{\bullet}$ constitutes a constructible complex of G -modules; that is, for each $i \in \mathbb{Z}$,

$$C^i = \bigoplus_{\mathbf{n} \in Z} C_{\mathbf{n}}^i(-\mathbf{n}) = \bigoplus_{\mathbf{g} \in G} C_{\mathbf{g}}^i = \bigoplus_{\mathbf{nq}} C_{\mathbf{nq}}^i$$

Proposition 7.14. *Fix a constructible family of complexes $\{C_{\mathbf{n}}^{\bullet}\}_{\mathbf{n} \in \mathbb{Z}}$ of Q -modules. For each $i \in \mathbb{Z}$ the cohomologies $\{H^i(C_{\mathbf{n}}^{\bullet})\}_{\mathbf{n} \in \mathbb{Z}}$ form a constructible family of Q -modules.*

Proof. This follows directly from Proposition 4.6 and the fact that kernels and cokernels of class $\mathfrak{X}$ morphisms are of class $\mathfrak{X}$ [Mil25, Proposition 4.30.1]. $\square$

7.3. Quasipolynomiality from constructibility.

Quasipolynomial consequences of constructibility of course pass through the numerical version, via an unassuming lemma, the tie that binds tameness to Presburger arithmetic.

Lemma 7.15. *Any constructible family is numerically constructible.*

Proof. Any semisimple subdivision subordinate to the module $\mathcal{M}$ associated to a constructible G -family has $\dim_{\mathbb{k}} M_{\mathbf{g}}$ constant for $\mathbf{g}$ in any constant region of G . $\square$

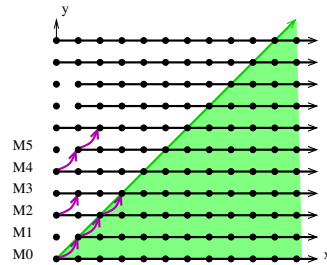
The following two examples show that the converse of Lemma 7.15 need not hold.

Example 7.16. Assume the situation of Example 3.6 with $G_+ = \mathbb{N} \times Q_+$, and further assume that the monomial ideal I is proper and nonzero. The family $\{M_n\}_{n \in \mathbb{Z}}$ is numerically constructible by Example 3.6, but there is no way to endow $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} M_n(-n)$ with a G -module structure in such a way that it is a constructible G -module. Indeed, homomorphisms between non-adjacent slices can't be isomorphisms, because any map $\mathbb{k}[x, y] \rightarrow \mathbb{k}[x, y]/I \oplus I$ of Q -modules must contain I in the kernel, and any Q -module map $\mathbb{k}[x, y]/I \oplus I \rightarrow \mathbb{k}[x, y]$ must vanish on the summand $\mathbb{k}[x, y]/I$, which is torsion. Consequently, any constant subdivision subordinate to any G -module structure on $\mathcal{M}$ must place each odd slice $\mathbb{k}[x, y]$ in a different region than every other odd slice. This prevents a finite constant subdivision.

Example 7.17. Fix $Q = \mathbb{Z}$ and $Q_+ = \mathbb{N}$. Let $G = \mathbb{Z} \times Q = \mathbb{Z}^2$ with positive cone G_+ (depicted in green) generated by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (the action of the latter depicted in purple). Let $\beta_0\beta_1\beta_2\ldots$ be a transcendental binary string. Then set

- $M_n = \mathbb{k}[\mathbb{N}]$ if $\beta_n = 0$;
- $M_n = \mathbb{k} \oplus \mathbb{k}\{1 + \mathbb{N}\}$ if $\beta_n = 1$, where $\mathbb{k} \cong \mathbb{k}[\mathbb{N}]/\mathbb{k}\{1 + \mathbb{N}\}$; and
- $M_n = 0$ if $n < 0$.

For example, with $\beta = 01000101100\ldots$:



Note that $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} M_n(-n)$ is a G -module, where every nonzero element of G_+ acts as a (shifted) inclusion on the $\mathbb{k}[\mathbb{N}]$ -free summands and as 0 on the torsion summands $\mathbb{k}$. Thus G_+ acts by translation along slices (as Q_+ does) or takes each Q -slice to a shifted

copy of that slice in higher levels. These shifts ensure that the aperiodic wiggling of the slice placements does not prevent the free summand of a lower slice from landing inside of the free summand of any higher slice.

This family $\{M_n\}_{n \in \mathbb{Z}}$ is numerically constructible, with two constant regions: the nonnegative quadrant of $G \cong \mathbb{Z}^2$ (where all of the vector-space dimensions are 1) and its complement (where all of the vector-space dimensions are 0).

However, the module $\mathcal{M}$ is not constructible because, for example, the locations of the vector space summands $\mathbb{k}$ occur at a subset of $\mathbb{Z} \times \{0\}$ whose transcendence prevents it from being semisimple. To be more precise, $\mathcal{M}$ dominates a finite constant subdivision with just three regions:

- the torsion degrees $\begin{bmatrix} n \\ 0 \end{bmatrix}$, each of which indexes a torsion summand $\mathbb{k}$ when $\beta_n = 1$;
- the torsion-free degrees, comprising all other nonnegative vectors in $\mathbb{Z}^2$; and
- the complement of the nonnegative quadrant, where $\mathcal{M}$ vanishes.

Thus $\mathcal{M}$ is tame but not semisimply constructible.

7.4. Constructible families of ideals.

Many main results in this paper assert that some operation on a constructible family yields another constructible family. That is, turning some crank preserves constructibility and hence can yield intricate output with piecewise quasipolynomial numerics. But where is the recursion to start? The purpose of this section is to provide rich sources of constructible families comprising—or in some cases, arising from—monomial ideals.

Definition 7.18. Let I and K be monomial ideals of $R = \mathbb{k}[Q_+]$.

1. The *Newton polyhedron* of I is the convex hull $\text{NP}(I)$ of $\{\mathbf{q} \in Q_+ \mid \mathbf{x}^{\mathbf{q}} \in I\}$.
2. The *integral closure* of I is ideal $\bar{I}$ whose monomials have some k th power in I^k :

$$\bar{I} = \mathbb{k}\{\mathbf{q} \in Q \mid \mathbf{x}^{k\mathbf{q}} \in I^k \text{ for some } k \in \mathbb{N}\}.$$

3. The *saturation* of I with respect to K is the ideal

$$I :_R K^\infty = \bigcup_{r \geq 0} I :_R K^r.$$

4. The *n th symbolic power* of I is

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \text{Min}(I)} (I^n R_{\mathfrak{p}} \cap R),$$

where $\text{Min}(-)$ denote the set of minimal primes.

5. R is *$\mathbb{Q}$ -Gorenstein* if some symbolic power of the ideal $\mathbb{k}\{Q_+^\circ\}$ is principal, where $Q_+^\circ \subset Q_+$ is the upset of lattice points interior to Q_+ . If $\mathbb{k}\{Q_+^\circ\}^{(r)} = \mathbb{k}[\mathbf{w} + Q_+]$, then the *multiplier ideal* of I is the monomial ideal $\mathcal{J}(I)$ such that

$$\mathbf{x}^{\mathbf{q}} \in \mathcal{J}(I) \Leftrightarrow \mathbf{q} + \mathbf{w}/r \in \text{NP}(I)^\circ,$$

the interior of the Newton polyhedron $\text{NP}(I)$.

Remark 7.19. The item numbers here refer to those in Definition 7.18.

2. It is nontrivial that $\bar{I}$ is an ideal, but in fact $\bar{I} = \mathbb{k}\{\text{NP}(I) \cap Q\}$ [SH06, §1.4].
3. The saturation $I^{\text{sat}} = I :_R \mathfrak{m}^\infty$ of I with respect to the homogeneous maximal ideal $\mathfrak{m} = \mathbb{k}\{Q_+ \setminus \{0\}\}$ of R is often called simply *the saturation of I* .
4. Write $\text{Ass}(-)$ for the associated primes and $\text{Ass}^\infty(I) = \bigcup_{n \geq 1} \text{Ass}(I^n)$. Then

$$I^{(n)} = I^n :_R K^\infty \text{ for the ideal } K = \bigcap_{\mathfrak{p} \in \text{Ass}^\infty(I) \setminus \text{Min}(I)} \mathfrak{p}.$$

The set $\text{Ass}^\infty(I)$ is finite by [Bro79] in much more generality than the monomial case here, but this finiteness is obvious over an affine semigroup ring, because the positive cone Q_+ has only finitely many faces.

5. $\mathcal{J}(I)$ is the characteristic-free combinatorial characterization from [HY03, Theorem 4.8] (see also [How01]); multiplier ideals have its origins and more general definitions in algebraic geometry [Laz04]. The $\mathbb{Q}$ -Gorenstein condition is satisfied when, for instance, Q_+ is smooth (Remark 2.9).

The following technical lemma is needed for the proof of Theorem 7.21.

Lemma 7.20. *Fix a smooth affine semigroup $N \subseteq \mathbb{Z}^n$ and a linear map $\lambda : \mathbb{Z}^n \rightarrow \mathbb{Z}$. Assume the hyperplane $H = \lambda^{-1}(1)$ intersects every extremal ray of the real cone $\mathbb{R}_{\geq 0}N$. There exists a smooth affine subsemigroup $N' \subseteq N$ and a positive integer r such that $N \cap rH \subseteq N'$. For all such N' there exists a finite partition $N = \bigcup_i (\mathbf{n}_i + N')$.*

Proof. Let $\mathbf{q}_1, \dots, \mathbf{q}_r$ be the generators of N and $\mathbf{p}_i = \mathbb{R}\mathbf{q}_i \cap H \in \mathbb{Q}N$. Fix $a_i, b_i \in \mathbb{Z}_{>0}$ such that $a_i \mathbf{p}_i = b_i \mathbf{q}_i$. The positive integer r can be taken to be any positive integer multiple of the product of the a_i . The submonoid N' is then generated by $r\mathbf{p}_1, \dots, r\mathbf{p}_t$. The last statement holds since the generators of N' are multiples of those of N . $\square$

The following theorem and proposition provide a wide range of examples of constructible families originating from ideals. Note in particular item 1, which allows the starting point to be any monomial ideal. As a matter of notation, assume the customary convention that $I^n = R$ whenever $n \leq 0$.

Theorem 7.21. *Fix an affine semigroup Q_+ with monoid algebra $R = \mathbb{k}[Q_+]$. Let $\{I_n\}_{n \in \mathbb{Z}}$ be a family of monomial ideals in R that is constructible over the Rees monoid of a nonzero monomial ideal $J \subseteq R$. Fix a monomial ideal $K \subseteq R$. Each of the following is a constructible family over the indicated Rees monoid G_+ .*

1. *The constant family $\{K\}_{n \in \mathbb{Z}}$, over the Rees monoid $G_+ = \mathbb{N} \times Q_+$.*
2. *The powers $\{I_n^{an+b}\}_{n \in \mathbb{Z}}$ for any fixed $a, b \in \mathbb{Z}$, over the Rees monoid G_+ of J^a .*
3. *The integral closures $\{\bar{I}_n\}_{n \in \mathbb{Z}}$, over the Rees monoid G_+ of J .*
4. *For $\mathbb{Q}$ -Gorenstein R , the multiplier ideals $\{\mathcal{J}(I_n)\}_{n \in \mathbb{Z}}$, over the Rees monoid of J .*

Proof. By Proposition 7.9, showing that each of these families $\{M_n\}_{n \in \mathbb{Z}}$ is constructible over the given Rees monoid G_+ is proved in two steps.

Step 1. $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} M_n(-n)$ is a G -module, where $G = \mathbb{Z}G_+$.

Step 2. Membership in $\{M_n\}_{n \in \mathbb{Z}}$ is Presburger (Definition 7.7).

Step 1 is immediate for items 1–2. For item 3 it follows from [SH06, Corollary 6.8.6]. For item 4, note that $J\mathcal{J}(I_n) \subseteq \mathcal{J}(JI_n) \subseteq \mathcal{J}(I_{n+1})$, since adding a point in $\text{NP}(J)$ with an interior point in $\text{NP}(I_n)$ leads to a point in the interior of $\text{NP}(JI_n)$.

The interesting part is Step 2. It follows for item 1 because K has a finite set of monomial generators, so focus on items 2–4. As usual when dealing with explicit Presburger formulas, identify $Q \cong \mathbb{Z}^d$.

For item 2, let $S \subset \mathbb{Z} \times \mathbb{Z}^d$ be the set of all $n\mathbf{q}$ with $\mathbf{q}$ a minimal monomial generator of I_n . By Lemma 7.8 S is Presburger definable. Theorem 2.10 and Definition 2.7 afford a finite partition $S = \bigcup_{i=1}^c (n_i \mathbf{q}_i + N_i)$ with $n_i \mathbf{q}_i \in \mathbb{Z} \times \mathbb{Z}^d$ and $N_i \subseteq \mathbb{Z} \times \mathbb{Z}^d$ smooth affine semigroups. It must be that $(\{0\} \times \mathbb{Z}^d) \cap N_i = \{0\} \times \{\mathbf{0}\}$, since each I_n has a finite number of generators. Therefore, by Lemma 7.20, assume that for each i all generators of N_i have the same first component. Let $F_i(n, \mathbf{q})$ be a Presburger formula that defines membership in N_i . Thus membership in $\{I_n^{an+b}\}_{n \in \mathbb{Z}}$ is defined by

$$F(n, \mathbf{q}) = (\exists m_1, \dots, m_c \in \mathbb{N}, \exists \mathbf{a}_1, \dots, \mathbf{a}_c \in \mathbb{Z}^d) \\ \left(\bigwedge_{i=1}^c F_i(m_i(n - n_i), \mathbf{a}_i) \wedge \sum_{i=1}^c m_i = an + b \wedge \mathbf{q} \succeq \sum_{i=1}^c (m_i \mathbf{q}_i + \mathbf{a}_i) \right),$$

where $\succeq$ denotes the partial order on Q .

For item 3 the set

$$\left\{ n\mathbf{q} \mid \exists k \in \mathbb{N}, \mathbf{p}_1, \dots, \mathbf{p}_k \in S, k\mathbf{q} \succeq \sum_{i=1}^k \mathbf{p}_i \right\},$$

with S as above, needs to be Presburger definable. And indeed it is, via the formula

$$G(n, \mathbf{q}) = (\exists k, m_1, \dots, m_c \in \mathbb{N}, \exists \mathbf{a}_1, \dots, \mathbf{a}_c \in \mathbb{Z}^d) \\ \left(\bigwedge_{i=1}^c F_i(m_i(n - n_i), \mathbf{a}_i) \wedge \sum_{i=1}^c m_i = k \wedge k\mathbf{q} \succeq \sum_{i=1}^c (m_i \mathbf{q}_i + \mathbf{a}_i) \right).$$

For item 4 it suffices to show that the set $\bigcup_{n \in \mathbb{Z}} \{ \{n\} \times (\text{NP}(I_n)^\circ \cap \mathbb{Z}^d) \}$ is Presburger definable. This set can be expressed as the intersection

$$\left\{ n\mathbf{q} \mid \exists k \in \mathbb{N}, \mathbf{p}_1, \dots, \mathbf{p}_k \in S, k\mathbf{q} + \mathbf{e}_i \succeq \sum_{i=1}^k \mathbf{p}_i \right\} \\ \cap \left\{ n\mathbf{q} \mid \exists k \in \mathbb{N}, \mathbf{p}_1, \dots, \mathbf{p}_k \in S, k\mathbf{q} - \mathbf{e}_i \succeq \sum_{i=1}^k \mathbf{p}_i \right\},$$

for $1 \leq i \leq d$. The proof now follows since each of these finitely many sets can be defined using a slight variation of the formula $G(n, \mathbf{q})$. $\square$

Proposition 7.22. *Assume Q_+ is an affine semigroup and set $R = \mathbb{k}[Q_+]$. Let $\{I_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ be a family of monomial ideals that is constructible over the Rees monoid of nonzero monomial ideals $J_1, \dots, J_k$. Fix a monomial ideal K . The following are constructible families over the specified Rees monoid.*

1. *The saturations $\{I_{\mathbf{n}} :_R K^\infty\}_{\mathbf{n} \in \mathbb{Z}^k}$, over the Rees monoid of $J_1, \dots, J_k$.*
2. *The colon ideals $\{I_{\mathbf{n}} :_R K^{\mathbf{a} \cdot \mathbf{n} + b}\}_{\mathbf{n} \in \mathbb{Z}^k}$ for fixed $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k$ and $b \in \mathbb{Z}$, over the Rees monoid of $J_1 :_R K^{a_1}, \dots, J_k :_R K^{a_k}$.*

Let $\{K_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ be another family of monomial ideals that is constructible over the Rees monoid of nonzero monomial ideals $H_1, \dots, H_k$. The following are constructible families over the Rees monoid of $J_1 H_1, \dots, J_k H_k$.

3. *The products $\{I_{\mathbf{n}} K_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$.*
4. *The sums $\{I_{\mathbf{n}} + K_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$.*
5. *The intersections $\{I_{\mathbf{n}} \cap K_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$.*
6. *The quotients $\{I_{\mathbf{n}}/K_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$, when $K_{\mathbf{n}} \subseteq I_{\mathbf{n}}$ for every $\mathbf{n}$.*

Let $\{Y_{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{Z}^\ell}$ be another family of monomial ideals that is constructible over the Rees monoid of nonzero monomial ideals $X_1, \dots, X_\ell$. The following are constructible families over the Rees monoid of $J_1, \dots, J_k, X_1, \dots, X_\ell$.

7. *The sums $\{I_{\mathbf{n}} + Y_{\mathbf{m}}\}_{\mathbf{n}\mathbf{m} \in \mathbb{Z}^{k+\ell}}$.*
8. *The products $\{I_{\mathbf{n}} Y_{\mathbf{m}}\}_{\mathbf{n}\mathbf{m} \in \mathbb{Z}^{k+\ell}}$.*
9. *The intersections $\{I_{\mathbf{n}} \cap Y_{\mathbf{m}}\}_{\mathbf{n}\mathbf{m} \in \mathbb{Z}^{k+\ell}}$.*

Proof. The argument as in Step 1 and Step 2 from Theorem 7.21 is straightforward. $\square$

Remark 7.23. Theorem 7.21.2 and Proposition 7.22.1 imply that for $a, b \in \mathbb{Z}$ the symbolic powers $\{I^{(an+b)}\}_{n \in \mathbb{Z}}$ form a constructible family over the Rees monoid of I .

Example 7.24. The different operations in Theorem 7.21 combine with Proposition 7.22 to produce intricate constructible families. For instance, when $R = \mathbb{k}[\mathbb{N}^d]$ and I, J, K are monomial ideals, the family $\{\overline{\mathcal{J}(I^n : J^{3n-2})} :_R K^\infty\}_{n \in \mathbb{N}}$ is constructible over the Rees monoid of $I :_R J^3$.

Example 7.25. Proposition 7.22.7–9 may be used to produce constructible families indexed by $\mathbb{Z}^k$ for $k > 1$, such as the family $\{I^{2n-1} \overline{J^{5m+1}} K^{(p)}\}_{(n,m,p) \in \mathbb{Z}^3}$ of monomial ideals, which is constructible over the Rees monoid of I^2, J^5, K .

Remark 7.26. In Lemma 9.25 it is shown that given a monomial ideal I and a constructible family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ of Q -modules, the submodules of the $M_{\mathbf{n}}$ consisting of the elements annihilated by I also form a constructible family.

Remark 7.27. There is a more general definition of multiplier ideals over arbitrary normal rings [dFH09], adapted to the monomial case in [HM18]. The conclusion of Theorem 7.21.4 still holds with that general definition.

7.5. Products of ideals and modules.

Additional constructible families of modules arise by multiplying any given family by ideals in a constructible family. The two main points in the proof are that (i) a module is constructible if and only if it admits an upset presentation with semisimple upsets, by the syzygy theorem [Mil25, Theorem 6.12.4], and (ii) Minkowski sums of semisimple sets (in particular, upsets) are semisimple.

The proposition making this precise requires an elementary lemma, which codifies a situation with two Rees monoids that occurs repeatedly in subsequent sections.

Lemma 7.28. *Fix Rees monoids G_+ and H_+ over Q with $G = Z \times Q$ and $H = Y \times Q$. The monoid $K_+ = G_+ \times_Q H_+$ defined by*

$$\mathbb{k}[K_+] = \mathbb{k}[G_+] \otimes_Q \mathbb{k}[H_+]$$

is a Rees monoid over Q , called the double Rees monoid of G_+ and H_+ over Q .

Proof. K_+ is a Rees monoid over Q because it is the submonoid $G_+ \times_Q H_+$ of the product $K = Z \times Y \times Q$ generated by the images of G_+ and H_+ under their embeddings via $G_+ \hookrightarrow G \cong Z \times \{\mathbf{0}\} \times Q \subseteq K$ and $H_+ \hookrightarrow H \cong \{\mathbf{0}\} \times Y \times Q \subseteq K$. $\square$

Proposition 7.29. *Fix Rees monoids G_+ and H_+ over Q with $G = Z \times Q$ and $H = Y \times Q$. If $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ is a constructible G -family of Q -modules and $\{I_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ is a constructible H -family of ideals in $\mathbb{k}[Q_+]$ with $I_{\mathbf{0}} = \mathbb{k}[Q_+]$, then over $K_+ = G_+ \times_Q H_+$*

1. *the products $\{I_{\mathbf{m}}M_{\mathbf{n}}\}_{\mathbf{nm} \in Z \times Y}$ form a constructible family of Q -modules, and*
2. *the quotients $\{M_{\mathbf{n}}/I_{\mathbf{m}}M_{\mathbf{n}}\}_{\mathbf{nm} \in Z \times Y}$ form a constructible family of Q -modules.*

Proof. Since each ideal $I_{\mathbf{m}}$ is the upset module $\mathbb{k}\{T_{\mathbf{m}}\} \subseteq \mathbb{k}\{Q\}$ for an upset $T_{\mathbf{m}} \subseteq Q$ (Definition 4.11), the H -module $\mathcal{I} = \bigoplus_{\mathbf{m} \in Y} I_{\mathbf{m}}(-\mathbf{m})$ associated to $\{I_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ is the upset H -module $\mathbb{k}\{T\}$ for a semisimple upset $T \subseteq H$ whose slices parallel to Q are $\{T_{\mathbf{m}}\}_{\mathbf{m} \in Y}$. If $U \subseteq G$ is any upset, then in $K = Z \times Y \times Q$, identifying T with $\{\mathbf{0}\} \times T$ and U with $U \times \{\mathbf{0}\}$, the Minkowski sum $T + U \subseteq K$ is an upset whose slice parallel to Q through $\mathbf{nm} \in Z \times Y$ is $T_{\mathbf{m}} + U_{\mathbf{n}}$. If U is semisimple then the Minkowski sum $T + U$ is semisimple because any Minkowski sum of semisimple sets is semisimple (the proof of Proposition 4.6 shows this, for example).

The family $\{I_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ of ideals is a subfamily of the *ambient family* $\{A_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ that has $A_{\mathbf{m}} = \mathbb{k}[Q_+]$ instead of $I_{\mathbf{m}}$ whenever $I_{\mathbf{m}} \neq 0$, so

$$\{A_{\mathbf{m}}\}_{\mathbf{m} \in Y} = \{\mathbb{k}[Q_+] \mid \mathbf{m} \in Y \text{ and } I_{\mathbf{m}} \neq 0\}.$$

We claim that the ambient family is constructible whenever $\{I_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ is. To see why, note first that any linear projection of a semisimple set is semisimple by Theorem 2.10, because the image of an affine semigroup under any linear map is finitely generated and hence another affine semigroup. Now apply this observation to the projections of H_+ and T to Y . It yields semisimple subsets $\pi_Y(H_+)$ and $\pi_Y(T)$ inside of Y . Let

$$H_+^A = \pi_Y(H_+) + Q_+ \quad \text{and} \quad S = \pi_Y(T) + Q_+$$

be the *ambient Rees monoid* constructed from H_+ and the *ambient upset* constructed from T . These are both semisimple, given that Q_+ is semisimple—again, because Minkowski sums of semisimple sets are semisimple—and $\mathbb{k}\{S\}$ is an upset H^A -module by construction. In fact, $\mathbb{k}\{S\}$ is the H^A -module associated to the ambient family.

The original family $\{I_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ and the ambient family $\{A_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ are two constructible families of ideals of $\mathbb{k}[Q_+]$, with $I_{\mathbf{m}} \subseteq A_{\mathbf{m}}$ for all $\mathbf{m} \in Y$. However, their Rees monoids need not coincide. (The reason: it is unlikely that $H_+^A + T \subseteq T$, because in most circumstances the ideals $I_{\mathbf{m}}$ decrease as $\mathbf{m}$ gets farther from the origin.) That said, hypothesis $I_0 = \mathbb{k}[Q_+]$ implies that

$$\mathbb{k}[H_+]_{\mathbf{m}} = \mathbb{k}[Q_+] \mathbb{k}[H_+]_{\mathbf{m}} = I_0 \mathbb{k}[H_+]_{\mathbf{m}} \subseteq I_{\mathbf{m}} \subseteq \{\mathbf{m}\} \times Q_+,$$

so the slice of H^+ parallel to Q over $\mathbf{m} \in Y$ is contained in the corresponding slice of H_+^A . It follows that $H_+^A \supseteq H_+$. Therefore the ambient family is an H -family, so it is constructible over the Rees monoid H_+ .

Going back to the first paragraph of the proof, think of U as an upset in an upset presentation (Definition 5.3) over $\mathbb{k}[G_+]$ of the associated G -module $\mathcal{M} = \bigoplus_{\mathbf{n} \in Z} M_{\mathbf{n}}(-\mathbf{n})$:

$$\bigoplus_{\rho=1}^r \mathbb{k}\{U^\rho\} \rightarrow \bigoplus_{\sigma=1}^s \mathbb{k}\{U^\sigma\}$$

which exists by the syzygy theorem for constructible modules [Mil25, Theorem 6.12.4] applied to the class $\mathfrak{X} = \text{semisimple}$ via Proposition 4.6 and Definition 4.9. The goal is to compare what happens when S and T are added to the upsets in this presentation.

Adding the ambient upset S yields an upset presentation

$$\bigoplus_{\rho=1}^r \mathbb{k}\{S + U^\rho\} \rightarrow \bigoplus_{\sigma=1}^s \mathbb{k}\{S + U^\sigma\}$$

of the *ambient* K -module $\mathcal{M}^A$ whose component in degree $\mathbf{m} \in Y$ is $M_{\mathbf{n}}$, independent of $\mathbf{m}$, by computing as in the first paragraph of the proof. The target in this presentation has the K -submodule $\bigoplus_{\sigma=1}^s \mathbb{k}\{T + U^\sigma\}$, whose image in $\mathcal{M}^A$ is $\mathcal{IM}$ since slicing this image through $\mathbf{nm} \in Z \times Y$ yields the image of $\bigoplus_{\sigma=1}^s \mathbb{k}\{T_{\mathbf{m}} + U_{\mathbf{n}}^\sigma\} \rightarrow M_{\mathbf{n}}$, which is $I_{\mathbf{m}}M_{\mathbf{n}}$. The induced K -module homomorphism $\bigoplus_{\sigma=1}^s \mathbb{k}\{T + U^\sigma\} \rightarrow \mathcal{M}^A$ is constructible because any semisimple subdivision of K subordinate to the given presentation of $\mathcal{M}^A$ becomes subordinate to the induced homomorphism upon common refinement with the subdivisions of K induced by the semisimple upsets $T + U^\sigma$ for all σ .

Constructibility of the products in item 1 follows because kernels and cokernels—and hence images—of constructible morphisms are constructible [Mil25, Proposition 4.30.1]. Constructibility of the products in item 2 now uses only the cokernel part. $\square$

Example 7.30. Since constructibility behaves well under direct sums [Mil25, Proposition 4.30], additional constructible families ensue. For example, if M is a constructible

module, then the family $\{M_n\}_{n \in \mathbb{N}}$ of Q -modules for

$$M_n = \mathcal{J}(I^n :_R J^n) / \overline{I^n} \oplus (I^{(n)} / I^n) \oplus (I^n :_R J^\infty) \oplus I^n M.$$

is constructible over the Rees monoid of I .

Remark 7.31. The proof of Proposition 7.29 uses that Minkowski sums of semisimple sets are semisimple. The statement and proof generalize to any class $\mathfrak{X}$ that is closed under Minkowski sums, such as when $\mathfrak{X}$ = semialgebraic or piecewise-linear.

8. FUNCTORS PRESERVING CONSTRUCTIBILITY OF FAMILIES

Applying the results in §6—particularly Theorems 6.2, 6.6, and 6.21—in the context of constructible families (Definition 7.5) leads to the conclusion that various functors applied to constructible families yield constructible families. This section makes this family-wise application precise, with subsections numbered as in §6.

Throughout this section, assume Q is a Presburger group (Definition 4.2).

8.1. Local cohomology.

Theorem 8.1. *Let $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ be a constructible family of Q -modules over a Rees monoid G_+ with $G = Z \times Q$ and fix a monomial ideal I in the monoid algebra $R = \mathbb{k}[Q_+]$. The family $\{H_I^i(M_{\mathbf{n}})\}_{\mathbf{n} \in Z}$ of local cohomology modules is constructible over G_+ for all i .*

Proof. Let $\mathcal{M} = \bigoplus_{\mathbf{n} \in Z} M_{\mathbf{n}}(-\mathbf{n})$. The ideal I is generated by monomials $\mathbf{x}^{\mathbf{q}_1}, \dots, \mathbf{x}^{\mathbf{q}_r}$ with $\mathbf{q}_i \in Q_+$. View these generators as lying in $\mathbb{k}[G_+ \cap (\{\mathbf{0}\} \times Q)] = \mathbb{k}[\{\mathbf{0}\} \times Q_+]$ by Definition 7.1, and use them to construct the Čech complex $\check{\mathcal{C}}_I^\bullet(\mathcal{M})$ to deduce that $H_I^i(\mathcal{M}) = \bigoplus_{\mathbf{n} \in Z} H_I^i(M_{\mathbf{n}})(-\mathbf{n})$ for all i . Since $\mathcal{M}$ is a constructible G -module by hypothesis, Theorem 6.2 implies that $\check{\mathcal{C}}_I^\bullet(\mathcal{M})$ is a constructible complex of G -modules and $H_I^\bullet(\mathcal{M})$ is a constructible G -module. $\square$

8.2. Tensor products and Tor.

The next definition is made for convenience, to avoid needlessly repeating the extensive hypotheses and notation of Lemma 8.4.

Definition 8.2. The families $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ and $\{N_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ of Q -modules over $G = Z \times Q$ and $H = Y \times Q$ in Lemma 8.4 form a *pair indexed by the double Rees monoid K_+* , with associated G -module $\mathcal{M} = \bigoplus_{\mathbf{n} \in Z} M_{\mathbf{n}}(-\mathbf{n})$ and H -module $\mathcal{N} = \bigoplus_{\mathbf{m} \in Y} N_{\mathbf{m}}(-\mathbf{m})$.

Recall the notion of semisimply flat module from Definition 5.9. The proof of Theorem 8.5 uses existence of semisimply flat resolutions—where the module in each homological degree is semisimply flat—over affine semigroup rings. The claim is of sufficient utility to separate it off into its own lemma.

Lemma 8.3. *If N is a localization of a noetherian or artinian module over an affine semigroup ring $\mathbb{k}[Q_+]$, then N admits a semisimply flat resolution and a semisimply injective resolution as a Q -module.*

Proof. The claim is immediate from Remarks 6.7 and 6.22. $\square$

Lemma 8.4. *Fix Rees monoids G_+ and H_+ over Q with $G = Z \times Q$ and $H = Y \times Q$. Let $K_+ = G_+ \times_Q H_+$ be their double Rees monoid over Q (Lemma 7.28). For any G -module $\mathcal{M}$ and H -module $\mathcal{N}$, if $\mathcal{M}^K = \mathcal{M} \otimes_G \mathbb{k}[K_+]$ there are natural isomorphisms*

$$\mathcal{M}^K = \mathcal{M} \otimes_Q \mathbb{k}[H_+] \quad \text{and} \quad \mathcal{M} \otimes_Q \mathcal{N} = \mathcal{M}^K \otimes_H \mathcal{N}.$$

In particular, for a G -family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ and an H -family $\{N_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ of Q -modules, if

$$\mathcal{M} = \bigoplus_{\mathbf{n} \in Z} M_{\mathbf{n}}(-\mathbf{n}) \quad \text{and} \quad \mathcal{N} = \bigoplus_{\mathbf{m} \in Y} N_{\mathbf{m}}(-\mathbf{m})$$

then

$$\mathcal{M} \otimes_Q \mathcal{N} = \bigoplus_{\mathbf{nm}} (M_{\mathbf{n}} \otimes_Q N_{\mathbf{m}})(-\mathbf{n}, -\mathbf{m}) = \mathcal{M}^K \otimes_H \mathcal{N}.$$

Proof. The isomorphisms are computed starting with

$$\begin{aligned} \mathcal{M}^K &= \mathcal{M} \otimes_G \mathbb{k}[K_+] \\ &= \mathcal{M} \otimes_G \mathbb{k}[G_+] \otimes_Q \mathbb{k}[H_+] \\ &= \mathcal{M} \otimes_Q \mathbb{k}[H_+], \end{aligned}$$

which implies that

$$\begin{aligned} \mathcal{M}^K \otimes_H \mathcal{N} &= \mathcal{M} \otimes_Q \mathbb{k}[H_+] \otimes_H \mathcal{N} \\ &= \mathcal{M} \otimes_Q \mathcal{N}. \end{aligned}$$

The final sentence follows because $\otimes_Q$ distributes over the direct sums in $\mathcal{M} \otimes_Q \mathcal{N}$. $\square$

Theorem 8.5. *Fix a pair of families $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ and $\{N_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ of Q -modules over $G = Z \times Q$ and $H = Y \times Q$ indexed by the double Rees monoid K_+ . Assume that*

- $\mathcal{M}$ is a constructible G -module;
- $\mathbb{k}[H_+]$ is flat as a module over $\mathbb{k}[Q_+]$, and
- $\mathcal{N}$ admits an H -flat resolution that is semisimply flat in degrees $i-1$, i , and $i+1$.

Then $\{\mathrm{Tor}_i^Q(M_{\mathbf{n}}, N_{\mathbf{m}})\}_{\mathbf{nm} \in Z \times Y}$ is a constructible family over K_+ . In particular,

$$\{\mathrm{Tor}_i^Q(M_{\mathbf{n}}, N)\}_{\mathbf{n} \in Z}$$

is constructible if the input family is constructible and the fixed input module is a localization of a noetherian or artinian module over an affine semigroup ring $\mathbb{k}[Q_+]$.

Proof. By Lemma 8.4, the functor $\mathcal{M} \mapsto \mathcal{M} \otimes_Q \mathcal{N}$ is a direct sum over $\mathbf{n}$ and $\mathbf{m}$ of functors $M_{\mathbf{n}} \mapsto M_{\mathbf{n}} \otimes_Q N_{\mathbf{m}}$. Each left derived functor of $\mathcal{M} \mapsto \mathcal{M} \otimes_Q \mathcal{N}$ is therefore a direct sum of the desired Tor modules. But Lemma 8.4 also says that $\mathcal{M} \mapsto \mathcal{M} \otimes_Q \mathcal{N}$ is isomorphic to the composite

$$\mathcal{M} \mapsto \mathcal{M} \otimes_Q k[H_+] \mapsto \mathcal{M} \otimes_Q k[H_+] \otimes_H \mathcal{N}$$

which is by definition

$$\mathcal{M} \mapsto \mathcal{M}^K \mapsto \mathcal{M}^K \otimes_H \mathcal{N}.$$

The flat hypothesis on H_+ implies that $\mathcal{M} \mapsto \mathcal{M}^K$ is exact, so the derived functor of the composite can be computed as the homology of

$$\mathcal{M}^K \otimes_H C_{\bullet}.$$

where $C_{\bullet}$ is the hypothesized H -flat resolution of $\mathcal{N}$.

The final goal, for the general claim about $\{\mathrm{Tor}_i^Q(M_{\mathbf{n}}, N_{\mathbf{m}})\}_{\mathbf{n}\mathbf{m} \in Z \times Y}$, is to show that $\mathcal{M}^K \otimes_H C_{\bullet}$ is a constructible complex of K -modules. First treat the case where G_+ is a free Rees monoid over Q . For any H -module $\mathcal{L}$, write $\mathcal{L}^K$ for the extension to K , so $\mathcal{L}^K = \mathcal{L} \otimes_H \mathbb{k}[K_+]$. Since G_+ is a free Rees monoid, $\mathbb{k}[K_+]$ is free as a $\mathbb{k}[H_+]$ -module. Hence $C_{\bullet}^K$ is flat and remains semisimply flat in homological degrees $i-1$, i , and $i+1$ because extension to K of a localization is a localization of $\mathbb{k}[K_+]$. The complex $\mathcal{M}^K \otimes_H C_{\bullet}$ is therefore constructible thanks to Theorem 6.6 with $\mathfrak{X} = \text{semisimple}$.

If the Rees monoid G_+ is not free over Q , then use Proposition 7.11 to replace G_+ with a free reduction G'_+ . Replacing G_+ with its reduction G'_+ has no effect on $\mathcal{M} \otimes_Q \mathcal{N}$ in Lemma 8.4, but it replaces the double Rees monoid K_+ with $K'_+ = G'_+ \times_Q H_+$ throughout Lemma 8.4, so $\mathcal{M} \otimes_Q \mathcal{N} = \mathcal{M}^{K'} \otimes_H \mathcal{N}$. Thus the desired result holds by the free case already proven.

For the “in particular” claim, note that $\mathbb{k}[H_+] = \mathbb{k}[Q_+]$ here, so Lemma 8.3 suffices to show that the hypotheses are satisfied. $\square$

Remark 8.6. In practice, the flatness hypotheses on H_+ and $\mathcal{N}$ are usually satisfied by taking $H_+ = Q_+$ and assuming that Q_+ is noetherian, so that $\mathcal{N}$ can be any finitely generated Q -module.

8.3. Hom and Ext.

Lemma 8.7. *Let G_+ and H_+ be Rees monoids over Q with $G = Z \times Q$ and $H = Y \times Q$. If $\mathcal{M}$ and $\mathcal{N}$ are the G - and H -modules associated to a pair of families $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ and $\{N_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ of Q -modules, and $\mathcal{N} \cong \mathcal{N}^{\vee\vee}$ (equivalently, $\dim_{\mathbb{k}} \mathcal{N}_{\mathbf{h}} < \infty$ for all $\mathbf{h} \in H$) then*

$$\bigoplus_{\mathbf{n}\mathbf{m}} \underline{\mathrm{Ext}}_Q^i(M_{\mathbf{n}}, N_{\mathbf{m}}) = \bigoplus_{\mathbf{n}\mathbf{m}} \mathrm{Tor}_i^Q(M_{\mathbf{n}}, N_{\mathbf{m}}^{\vee})^{\vee}$$

Proof. Swap the roles of N and $N^{\vee}$ in Corollary 6.12, as allowed by Lemma 6.13. $\square$

Theorem 8.8. *Fix a pair of families $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ and $\{N_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ of Q -modules over $G = Z \times Q$ and $H = Y \times Q$ indexed by the double Rees monoid K_+ , with associated G -module $\mathcal{M}$ and H -module $\mathcal{N}$. Assume that one of the following two scenarios is in effect.*

1. *$\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ is constructible, $\mathbb{k}[H_+]$ is flat over $\mathbb{k}[Q_+]$, and some injective resolution of $\mathcal{N} = \mathcal{N}^{\vee\vee}$ is semisimply injective in cohomological degrees $i-1$, i , and $i+1$.*
2. *$\{N_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ is a constructible family, $\mathbb{k}[G_+]$ is flat over $\mathbb{k}[Q_+]$, and some flat resolution of $\mathcal{M}$ is semisimply flat in homological degrees $i-1$, i , and $i+1$.*

Then $\{\underline{\mathrm{Ext}}_Q^i(M_{-\mathbf{n}}, N_{\mathbf{m}})\}_{\mathbf{n}, \mathbf{m} \in Z \times Y}$ is a constructible family over K_+ . In particular,

$$\{\underline{\mathrm{Ext}}_Q^i(M_{\mathbf{n}}, N)\}_{\mathbf{n} \in Z} \quad \text{and} \quad \{\underline{\mathrm{Ext}}_Q^i(M, N_{\mathbf{m}})\}_{\mathbf{m} \in Y}$$

are constructible if the input family is constructible and the fixed input module is a localization of a noetherian or artinian module over an affine semigroup ring $\mathbb{k}[Q_+]$.

Proof. The direct sum $\bigoplus_{\mathbf{n}, \mathbf{m}} \mathrm{Tor}_i^Q(M_{\mathbf{n}}, N_{\mathbf{m}})(-\mathbf{n}, -\mathbf{m})$ is a constructible K -module by Theorem 8.5 combined with symmetry of Tor , as long as the itemized hypotheses of Theorem 8.5 are satisfied verbatim or they are satisfied with $\mathcal{M}$ and $\mathcal{N}$ as well as G and H swapped. The latter option is precisely the second scenario here. The Matlis dual of this Tor direct sum is

$$\bigoplus_{\mathbf{n}, \mathbf{m}} \mathrm{Tor}_i^Q(M_{-\mathbf{n}}, N_{-\mathbf{m}})^{\vee}(-\mathbf{n}, -\mathbf{m}),$$

which remains constructible by Lemma 6.13 as long as the hypotheses remain in force unchanged. Replacing $\mathcal{N}$ with $\mathcal{N}^{\vee}$ (the Matlis dual here is over H , although it hardly matters, given the degree-by-degree phrasing of Definition 6.8), yields

$$\bigoplus_{\mathbf{n}, \mathbf{m}} \mathrm{Tor}_i^Q(M_{-\mathbf{n}}, N_{\mathbf{m}}^{\vee})^{\vee}(-\mathbf{n}, -\mathbf{m}),$$

and this altered version is again constructible by Theorem 8.5, either under the second scenario unchanged, or under the hypotheses of Theorem 8.5 as written but with $\mathcal{N}^{\vee}$ in place of $\mathcal{N}$. Given that $\mathcal{N} = \mathcal{N}^{\vee\vee}$, the semisimply flat hypothesis on the resolution of $\mathcal{N}^{\vee}$ from Theorem 8.5 is equivalent to the semisimply injective hypothesis on the resolution of $\mathcal{N}$ in the first scenario by Lemma 6.13. Applying Lemma 8.7 to this last Tor direct sum yields the desired result.

The hypotheses for the “in particular” claim are set up so that the fixed input module has a resolution that is semisimply flat or injective by Lemma 8.3 or its Matlis dual (see Lemma 6.13 and Remark 6.14). For $\underline{\mathrm{Ext}}_Q^i(M_{\mathbf{n}}, N)$ the injective resolution of N is used for scenario 1; for $\underline{\mathrm{Ext}}_Q^i(M, N_{\mathbf{m}})$ the flat resolution of M is used for scenario 2. $\square$

8.4. Functorial quasipolynomiality.

Combining the piecewise quasipolynomial behavior of numerically constructible families (§3.2) with the functorial preservation of constructibility (§8.1–8.3) yields prototypical piecewise quasipolynomiality results as corollaries.

Recall the length, max, and min functions ℓ_V , $\max_{\lambda,V}$ and $\min_{\lambda,V}$ defined for a family $\{V_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ of Q -graded vector spaces in Theorem 3.11. To simplify the subsequent corollaries, as well as results in §9, use the following two notational conventions.

Convention 8.9. Domains of functions $\mathbf{n} \rightarrow \mathbb{Z}$ shown to be quasipolynomial are not explicitly indicated. For example, if the function is $\ell(M_{\mathbf{n}})$, the conclusion implicitly refers to those $\mathbf{n}$ for which the length is finite; and if the function is $\max\{|\mathbf{a}| \mid (M_{\mathbf{n}})_{\mathbf{a}} \neq 0\}$, the conclusion implicitly refers to those $\mathbf{n}$ for which the maximum exists.

Definition 8.10. The integers $\{m_{\mathbf{n}}\}_{\mathbf{n} \in T}$ indexed by a subset $T \subseteq \mathbb{Z}^k$ are *piecewise quasipolynomial* or *piecewise quasilinear* if the function $g : T \rightarrow \mathbb{Z}$ with $g(\mathbf{n}) = m_{\mathbf{n}}$ is piecewise quasipolynomial or piecewise quasilinear (Definition 3.8). Recall that if $k = 1$ and $T \subseteq \mathbb{N}$, this means $g(n)$ coincides with a quasipolynomial for $n \gg 0$ (Remark 3.9).

Corollary 8.11. Fix a monomial ideal I in the monoid algebra $R = \mathbb{k}[Q_+]$ and a constructible family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ of Q -modules over a Rees monoid G_+ with $G = Z \times Q$. If $\{V_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ is the family of local cohomology modules $V_{\mathbf{n}} = H_I^i(M_{\mathbf{n}})$, then the length $\ell_V : Z \rightarrow \mathbb{N}$ is piecewise quasipolynomial of degree at most $\text{rank } Q$. Moreover, the functions $\max_{\lambda,V}$ and $\min_{\lambda,V}$ are piecewise quasilinear for any linear function $\lambda : Q \rightarrow \mathbb{Z}$.

Proof. Apply Theorem 3.11 to the conclusion of Theorem 8.1 via Lemma 7.15. $\square$

Corollary 8.12. Fix Rees monoids G_+ and H_+ over Q with $G = Z \times Q$ and $H = Y \times Q$. Let $\{M_{\mathbf{n}}\}_{\mathbf{n} \in Z}$ and $\{N_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ be a G -family and an H -family of Q -modules. Assume $\{M_{\mathbf{n}}\}$ is constructible, H_+ is flat over Q , and $\mathcal{N} = \bigoplus_{\mathbf{m} \in Y} N_{\mathbf{m}}(-\mathbf{m})$ admits an H -flat resolution that is semisimply flat in degrees $i-1$, i , and $i+1$. Let $V_{\mathbf{nm}} = \text{Tor}_i^Q(M_{\mathbf{n}}, N_{\mathbf{m}})$.

1. The length $\ell_V : Z \times Y \rightarrow \mathbb{N}$ is piecewise quasipolynomial of degree at most $\text{rank } Q$.
2. If $\lambda : Q \rightarrow \mathbb{Z}$ is linear, then $\max_{\lambda,V}$ and $\min_{\lambda,V}$ are piecewise quasilinear on $Z \times Y$.

In particular, if N is a localization of a noetherian or artinian module over an affine semigroup ring $\mathbb{k}[Q_+]$ and $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ is a constructible family of Q -modules, then

$$\ell(\text{Tor}_i^Q(M_{\mathbf{n}}, N)) \quad \text{and} \quad \max_{\lambda}(\text{Tor}_i^Q(M_{\mathbf{n}}, N)) \quad \text{and} \quad \min_{\lambda}(\text{Tor}_i^Q(M_{\mathbf{n}}, N))$$

are piecewise quasipolynomial on $\mathbb{Z}^k$ of degrees at most $\text{rank } Q$, 1, and 1, respectively.

Proof. Apply Theorem 3.11 to the conclusion of Theorem 8.5 via Lemma 7.15. $\square$

Corollary 8.13. *Fix Rees monoids G_+ and H_+ over Q with $G = Z \times Q$ and $H = Y \times Q$. Let $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ be a G -family and $\{N_{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{Z}^\ell}$ an H -family of Q -modules. Assume that one of the following two scenarios is in effect.*

1. *$\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ is constructible, H_+ is flat over Q , and $\mathcal{N} = \bigoplus_{\mathbf{m} \in Y} N_{\mathbf{m}}(-\mathbf{m})$ has a $\mathbb{k}[H_+]$ -injective resolution that is semisimply injective in degrees $i-1$, i , and $i+1$.*
2. *$\{N_{\mathbf{m}}\}_{\mathbf{m} \in Y}$ is constructible, G_+ is flat over Q , and $\mathcal{M} = \bigoplus_{\mathbf{n} \in \mathbb{Z}^k} M_{\mathbf{n}}(-\mathbf{n})$ has a $\mathbb{k}[G_+]$ -flat resolution that is semisimply flat in degrees $i-1$, i , and $i+1$.*

Let $V_{\mathbf{nm}} = \underline{\text{Ext}}_Q^i(M_{\mathbf{n}}, N_{\mathbf{m}})$ for $\mathbf{nm} \in Z \times Y$.

1. *The length $\ell_V : Z \times Y \rightarrow \mathbb{N}$ is piecewise quasipolynomial of degree at most $\text{rank } Q$.*
2. *If $\lambda : Q \rightarrow \mathbb{Z}$ is linear, then $\max_{\lambda, V}$ and $\min_{\lambda, V}$ are piecewise quasilinear on $Z \times Y$.*

In particular, if N is a localization of a noetherian or artinian module over an affine semigroup ring $\mathbb{k}[Q_+]$ and $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ is a constructible family of Q -modules, then

$$\ell(\underline{\text{Ext}}_Q^i(M_{\mathbf{n}}, N)) \quad \text{and} \quad \max_{\lambda}(\underline{\text{Ext}}_Q^i(M_{\mathbf{n}}, N)) \quad \text{and} \quad \min_{\lambda}(\underline{\text{Ext}}_Q^i(M_{\mathbf{n}}, N))$$

are piecewise quasipolynomial on $\mathbb{Z}^k$ of degrees $\leq \text{rank } Q$, 1, and 1, respectively, as are

$$\ell(\underline{\text{Ext}}_Q^i(N, M_{\mathbf{n}})) \quad \text{and} \quad \max_{\lambda}(\underline{\text{Ext}}_Q^i(N, M_{\mathbf{n}})) \quad \text{and} \quad \min_{\lambda}(\underline{\text{Ext}}_Q^i(N, M_{\mathbf{n}})).$$

Proof. Apply Theorem 3.11 to the conclusion of Theorem 8.8 via Lemma 7.15. $\square$

Remark 8.14. The piecewise polynomial conclusions of all three corollaries in this section remain valid for any linear truncation (Remark 3.12) of the families of local cohomology, Tor, or $\underline{\text{Ext}}$ modules. This can be seen by inserting Remark 3.12 between the applications of Lemma 7.15 and Theorem 3.11 in the proofs of the corollaries.

9. UBIQUITY OF QUASIPOLYNOMIALS

This section covers applications of our theory, showcasing how it can be used to prove that fundamental homological invariants behave quasipolynomially. Some of the conclusions here greatly extend, in the context of families of multigraded modules over affine semigroup rings, several results in the literature.

Convention 9.1. Throughout this section, in addition to Convention 8.9 and Definition 8.10, assume the following.

1. Q_+ is an affine semigroup (Definition 2.8) with $Q = \mathbb{Z}Q_+ \cong \mathbb{Z}^d$. Set $R = \mathbb{k}[Q_+]$ and $\mathfrak{m} = \mathbb{k}\{Q_+ \setminus \{\mathbf{0}\}\}$ its homogeneous maximal ideal.
2. A family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ of Q -modules is *constructible* if it is so over some Rees monoid G_+ over Q with $G = \mathbb{Z}^k \times Q$ as in Definition 7.5.

9.1. Local cohomology, $\mathbf{a}$ -invariants, and regularity.

Lemma 9.2. *Let $\{M_{\mathbf{n}}^1\}_{\mathbf{n} \in \mathbb{Z}^k}, \dots, \{M_{\mathbf{n}}^r\}_{\mathbf{n} \in \mathbb{Z}^k}$ be constructible families of Q -modules. The functions $\mathbf{n} \mapsto \max\{i \mid M_{\mathbf{n}}^i \neq 0\}$ and $\mathbf{n} \mapsto \min\{i \mid M_{\mathbf{n}}^i \neq 0\}$ are piecewise quasiconstant (piecewise quasipolynomial of degree 0).*

Proof. By Theorem 2.10 it suffices to show that for each j the sets

$$A_j = \{\mathbf{n} \in \mathbb{Z}^k \mid j = \max\{i \mid M_{\mathbf{n}}^i \neq 0\}\}$$

and $B_j = \{\mathbf{n} \in \mathbb{Z}^k \mid j = \min\{i \mid M_{\mathbf{n}}^i \neq 0\}\}$

are semisimple. For each i some Presburger formula $F_i(\mathbf{n})$ defines the set $\{\mathbf{n} \in \mathbb{Z}^k \mid M_{\mathbf{n}}^i \neq 0\}$. Thus A_j is defined by $F_j(\mathbf{n}) \wedge \bigwedge_{i>j} \neg F_i(\mathbf{n})$ and B_j by $F_j(\mathbf{n}) \wedge \bigwedge_{i<j} \neg F_i(\mathbf{n})$. $\square$

Remark 9.3. If $k = 1$, piecewise quasiconstant is equivalent to eventually periodic.

Theorem 9.4. *Let $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ be a constructible family of Q -modules.*

1. *For any monomial ideal I and $a, b, c, d \in \mathbb{Z}$ the lengths*

$$\ell\left(\bigoplus_{a|\mathbf{n}|+b \leq |\mathbf{a}| \leq c|\mathbf{n}|+d} H_I^i(M_{\mathbf{n}})_{\mathbf{a}}\right) \quad \text{and} \quad \ell(H_{\mathfrak{m}}^i(M_{\mathbf{n}})_{\geq a|\mathbf{n}|+b})$$

are piecewise quasipolynomial of degree at most d .

2. *The $\mathbf{a}$ -invariants $a_i(M_{\mathbf{n}}) = \max\{|\mathbf{a}| \mid H_{\mathfrak{m}}^i(M_{\mathbf{n}})_{\mathbf{a}} \neq 0\}$ are piecewise quasilinear.*
3. *The Castelnuovo–Mumford regularities $\text{reg}(M_{\mathbf{n}}) = \max\{a_i(M_{\mathbf{n}}) + i\}$ are piecewise quasilinear.*
4. *The depths $\text{depth}(M_{\mathbf{n}}) = \min\{i \in \mathbb{N} \mid H_{\mathfrak{m}}^i(M_{\mathbf{n}}) \neq 0\}$ and dimensions $\dim(M_{\mathbf{n}}) = \max\{i \in \mathbb{N} \mid H_{\mathfrak{m}}^i(M_{\mathbf{n}}) \neq 0\}$ are piecewise quasiconstant.*

Proof. Part 1 follows from Corollary 8.11 and Remark 3.12; Part 2 from Corollary 8.11, and Part 3 from Part 2. Part 4 follows from Lemma 9.2 applied to the families $\{H_{\mathfrak{m}}^i(M_{\mathbf{n}})\}_{\mathbf{n} \in \mathbb{Z}^k}$. $\square$

Remark 9.5. Theorem 9.4.1 generalizes a result by Dao and Montaña [DM19, Theorem 3.8], which states that for a monomial ideal $I \subseteq R = \mathbb{k}[\mathbb{N}^d]$, the lengths of $H_{\mathfrak{m}}^i(R/I^n)$ are eventually quasipolynomial if they are finite for $n \gg 0$. The proof there use Takayama's generalization [Tak05] of Hochster's formula [Hoc75] to describe the constant regions in the local cohomology of monomial ideals explicitly by indexing them with labels that are subcomplexes of a fixed simplicial complex on d vertices. That reduces the characterization of constant regions in local cohomology to ideal membership in powers of monomial ideals, which they directly prove to be semisimple. Constructible module theory here allows conclusions on the semisimplicity of constant regions without identifying the regions explicitly, and with arbitrary constructible input.

Remark 9.6. Using the ideas in [DM19, Theorem 3.8], Das showed that for monomial ideals $I_1, \dots, I_k$ in $R = \mathbb{k}[\mathbb{N}^d]$ the lengths $\ell(H_{\mathfrak{m}}^0(R/(I_1^{n_1} \cdots I_k^{n_k})))$ are quasipolynomial for $n_1, \dots, n_k \gg 0$. Theorem 9.4.1 generalizes this result to constructible families. In particular, by Proposition 7.22.6 and 7.22.8, our theorem applies to $R/I_{1,n_1} \cdots I_{k,n_k}$, where each $\{I_{i,n}\}_{n \in \mathbb{Z}}$ is a constructible family of ideals.

Remark 9.7. Regularities of powers of homogeneous ideals in arbitrary standard graded noetherian rings are known to eventually agree with a linear function [CHT99, Kod00, TW05]. Theorem 9.4.3 shows that, over affine semigroup rings, this linear phenomenon occurs in the much larger class of constructible families of modules.

Remark 9.8. When $R = \mathbb{k}[\mathbb{N}^d]$ and $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} M_n(-n)$ is noetherian over an underlying Rees monoid, classical methods show that $\text{depth}(M_n)$ is periodic for $n \gg 0$ [HH05]: it holds for finitely generated graded modules over a noetherian positively graded algebra whose zeroth component is either a local ring or a standard graded ring over a field. In the multigraded setting, Theorem 9.4.4 applies more generally when $\mathcal{M}$ only needs to be constructible (see Remark 4.12). In [NT19] the authors show that any eventually periodic function can be realized as the depths of symbolic powers of a homogeneous (not necessarily monomial) ideal in a polynomial ring.

Example 9.9. To indicate an arbitrary instance of the kinds of composite operations that still yield controlled numerical growth, let I, J, K be monomial ideals in the polynomial ring $R = \mathbb{k}[\mathbb{N}^d]$. The regularities and depths of integral closures of saturations of multiplier ideals of colons of powers exhibit controlled numerical growth:

1. $\text{reg}\left(\overline{\mathcal{J}(I^n :_R J^{3n-2}) :_R K^\infty}\right)$ is quasilinear, and
2. $\text{depth}\left(R/\overline{\mathcal{J}(I^n :_R J^{3n-2}) :_R K^\infty}\right)$ is periodic for $n \gg 0$,

as do lengths of local cohomology of quotients mod integral closures of colons of powers:

3. $\ell(H_{\langle x, y \rangle}^i(R/\overline{I^n :_R \mathfrak{m}^n})_{-n \leq |\mathbf{a}| \leq n})$ is quasipolynomial of degree at most d for $n \gg 0$.

Note that the ideals in items 1 and 2 do not necessarily form graded families.

9.2. Tor and Betti numbers.

Specializing the tensor products, lengths, max, and min in Corollary 8.12 yields conclusions about minimal generators and higher Betti numbers. Recall Convention 9.1.

Theorem 9.10. *Let $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ be a constructible family of Q -modules.*

1. *If N is a localization of a noetherian or artinian R -module, then as functions of $\mathbf{n} \in \mathbb{Z}^k$, the length and number of generators*

$$\ell(\text{Tor}_i^Q(M_{\mathbf{n}}, N)) \quad \text{and} \quad \mu(\text{Tor}_i^Q(M_{\mathbf{n}}, N))$$

are piecewise quasipolynomial of degree at most d .

2. *The Betti numbers $\beta_i(M_{\mathbf{n}}) = \dim_{\mathbb{k}} \text{Tor}_i^Q(M_{\mathbf{n}}, \mathbb{k})$ are piecewise quasipolynomial of degree at most d . This includes the number of generators $\mu(M_{\mathbf{n}}) = \beta_0(M_{\mathbf{n}})$.*

Proof. The length conclusion in part 1 is part of the fixed- N case of Corollary 8.12. Part 2 follows from the length conclusion in part 1 with input module $N = \mathbb{k}$. The number of generators conclusion in part 1 follows from part 2 and the constructibility of $\mathrm{Tor}_i^Q(M_{\mathbf{n}}, N)$ from Theorem 8.5 with fixed input module N . $\square$

Remark 9.11. The proof of Theorem 9.10 feels routine, but note that it rests heavily on the recursive strength of functorial preservation of constructibility from §8.

Remark 9.12. Theorem 9.10.2 implies that for a monomial ideal I , the *symbolic defects* $\mu(I^{(n)}/I^n)$ are quasipolynomial for $n \gg 0$. This recovers [DG20, Theorem 2.4] in the case of monomial ideals in affine semigroup rings.

Remark 9.13. Resolutions of Rees algebras as modules over polynomial rings have been used to conclude polynomial growth of $\mathrm{Tor}_i(M/I^n M, N/J^m N)$ in a general commutative local noetherian setting, when these modules all have finite length [The03]. The relevant resolutions in our setting would be multigraded by the monoid underlying the Rees algebra but not finely graded over the polynomial ring. It might be possible to extend the constructibility methods here to that setting, but such an extension lies beyond the scope of the current initial investigation.

Definition 9.14. A sequence of ideals $\{I_n\}_{n \in \mathbb{N}}$ is a *graded family* if $I_0 = R$ and $I_n I_m \subseteq I_{n+m}$ for every $n, m \in \mathbb{N}$. A graded family is *noetherian* if the graded algebra $\bigoplus_{n \in \mathbb{N}} I_n(-n)$ is noetherian.

Example 9.15. Let I be a monomial ideal of R . By [SH06, Corollary 9.2.1], which is stated for any ideal in any analytically unramified Noetherian local ring, the integral closures $\{\bar{I}^n\}_{n \in \mathbb{N}}$ form a noetherian graded family. By [HHT07, Theorem 3.2] so do the saturations $\{I^n :_R K^\infty\}_{n \in \mathbb{N}}$ of a monomial ideal I with respect to a monomial ideal K . In particular, the symbolic powers $\{I^{(n)}\}_{n \in \mathbb{N}}$ of a monomial ideal I form a noetherian graded family. When Q_+ is smooth, the direct sum of multiplier ideals $\bigoplus_{n \in \mathbb{N}} \mathcal{J}(I^n)(-n)$ is a noetherian module over the Rees monoid of I [SH06, Proposition 18.2.4], which is stated for ideals in any regular domain.

Example 9.16. Fix monomial ideals I and J in $R = \mathbb{k}[\mathbb{N}^d]$, and let M be a noetherian R -module. The number of generators $\mu(\mathrm{Tor}_i^Q(R/\mathcal{J}(I^n :_R J^n), M))$ and the lengths $\ell(\mathrm{Tor}_i^Q(R/\mathcal{J}(I^n :_R J^n), M))$ are quasipolynomials of degree at most d for $n \gg 0$.

9.3. Ext and Bass numbers.

Definition 9.17. The *Bass number* $\mu_i(\mathfrak{p}, M)$ of a Q -module M at the monomial prime ideal $\mathfrak{p}$ in cohomological degree i is the number of indecomposable summands isomorphic to a shift of the multigraded injective hull of $R/\mathfrak{p}$ appearing at stage i in any minimal graded injective resolution of M .

Remark 9.18. Definition 9.17 of Bass numbers agrees with the usual (ungraded) Bass numbers from commutative algebra [GW78, Theorems 1.2.3 and 1.3.4]. These numbers also determine the ungraded Bass numbers at all primes [GW78, Theorem 1.2.3]. For this reason we state our results on Bass numbers only for monomial prime ideals.

Lemma 9.19. *If $\mathfrak{p}$ is a monomial prime then $\mathfrak{p} = \mathbb{k}[Q_+ \setminus F]$ for some face F of Q_+ (Definition 5.16). The Bass numbers of M at $\mathfrak{p}$ can be computed as*

$$\mu_i(\mathfrak{p}, M) = \text{rank}_{(R/\mathfrak{p})_F} \underline{\text{Ext}}_Q^i(R/\mathfrak{p}, M)_F$$

where $(-)_F$ denotes localization as in Definition 5.7 and $\text{rank}_{(R/\mathfrak{p})_F}$ is the rank as a free module over $(R/\mathfrak{p})_F$.

Proof. See [GW78, Theorem 1.1.4]. □

Theorem 9.20. *Let $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ be a constructible family of Q -modules.*

1. *If N is a localization of a noetherian or artinian R -module, then as functions of $\mathbf{n} \in \mathbb{Z}^k$, the lengths*

$$\ell(\underline{\text{Ext}}_Q^i(M_{\mathbf{n}}, N)) \quad \text{and} \quad \ell(\underline{\text{Ext}}_Q^i(N, M_{\mathbf{n}}))$$

and numbers of generators

$$\mu(\underline{\text{Ext}}_Q^i(M_{\mathbf{n}}, N)) \quad \text{and} \quad \mu(\underline{\text{Ext}}_Q^i(N, M_{\mathbf{n}}))$$

are piecewise quasipolynomials of degree at most d .

2. *If $\mathfrak{p}$ is a monomial prime ideal, then the Bass numbers $\mu_i(\mathfrak{p}, M_{\mathbf{n}})$ are piecewise quasipolynomial of degree at most $d - \dim F$.*

Proof. The length conclusion in part 1 is part of the fixed- N case of Corollary 8.13. The number of generators conclusion in part 1 follows from Theorem 9.10.2 and the constructibility of the relevant families of $\underline{\text{Ext}}$ modules from Theorem 8.8 with fixed input module N . For part 2, the family $\underline{\text{Ext}}_Q^i(R/\mathfrak{p}, M_{\mathbf{n}})$ is constructible by Theorem 8.8 with $N = R/\mathfrak{p}$. Since localization preserves constructibility by Lemma 5.12, the family $\underline{\text{Ext}}_Q^i(R/\mathfrak{p}, M_{\mathbf{n}})_F$ is constructible. Let $F^\perp \subseteq Q$ be a complement to $\mathbb{Z}F$, meaning that $F^\perp \cap \mathbb{Z}F = \{\mathbf{0}\}$ and $\text{rank } F^\perp + \text{rank } F = d = \text{rank } Q$. Then

- the restriction $\{E_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ of the family $\underline{\text{Ext}}_Q^i(R/\mathfrak{p}, M_{\mathbf{n}})_F$ to $F^\perp$ is constructible over the Presburger group $F^\perp \cap Q$ because the intersection of any semisimple set with a subgroup of Q is semisimple; and
- for each $\mathbf{n} \in \mathbb{Z}^k$, the rank of $\underline{\text{Ext}}_Q^i(R/\mathfrak{p}, M_{\mathbf{n}})_F$ over $(R/\mathfrak{p})_F$ equals the dimension of $E_{\mathbf{n}}$ as a vector space over $\mathbb{k}$ because $\underline{\text{Ext}}_Q^i(R/\mathfrak{p}, M_{\mathbf{n}})_F$ is free over $(R/\mathfrak{p})_F$.

These vector space dimensions $\dim_{\mathbb{k}} E_{\mathbf{n}}$ are therefore the desired Bass numbers by Lemma 9.19, and they are piecewise quasipolynomial of degree at most $n - \dim F$ by Theorem 3.11.1 via Lemma 7.15, because that is the rank of $F^\perp$. □

Remark 9.21. Theorems 9.10.1 and 9.20.1, together with Proposition 7.29, extend to constructible modules the results in [Kod93, The02] about polynomial behavior of lengths and number of generators of Tor and Ext.

Example 9.22. For any monomial ideal I and monomial prime ideal $\mathfrak{p}$, the Bass numbers $\mu_i(\mathfrak{p}, \overline{I^n}/I^{5n-2})$ are quasipolynomials of degree at most d for $n \gg 0$.

9.4. v -invariants.

Definition 9.23. The v -invariant of a monomial ideal I at a prime $\mathfrak{p} \in \text{Ass}(R/I)$ is

$$v_{\mathfrak{p}}(I) = \min\{|\mathbf{u}| \mid \mathfrak{p} = I :_R f \text{ for some } f \in R_{\mathbf{u}}\}.$$

Remark 9.24. In [Con24], [FS23] it was shown that for any $\mathfrak{p} \in \text{Ass}^\infty(I)$ the sequence $v_{\mathfrak{p}}(I^n)$ agrees with a linear polynomial for $n \gg 0$ (their results work more generally for homogeneous ideals in noetherian standard graded domains). The present goal is to extend this result to constructible families of Q -modules.

Lemma 9.25. *Given a constructible family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}}$ of Q -modules and a monomial ideal $I \subseteq R$, the family $\{0 :_{M_{\mathbf{n}}} I\}_{\mathbf{n} \in \mathbb{Z}}$ of Q -modules is constructible.*

Proof. Let G_+ be an underlying Rees monoid of $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}}$, so the associated G -module $\mathcal{M} = \bigoplus_{\mathbf{n} \in \mathbb{Z}} M_{\mathbf{n}}(-\mathbf{n})$ is constructible. By Convention 9.1.1, the ideal I is finitely generated by monomials $\mathbf{x}^{\mathbf{q}_1}, \dots, \mathbf{x}^{\mathbf{q}_r}$. Multiplication by $\mathbf{x}^{\mathbf{q}_i}$ induces an endomorphism $\varphi_i : \mathcal{M} \rightarrow \mathcal{M}$ that is constructible by Proposition 7.29. The kernel therefore yields a constructible submodule $\ker(\varphi_i) \subseteq \mathcal{M}$ [Mil25, Proposition 4.30.1]. The common refinement of any constant subdivisions subordinate to these kernels is subordinate to their intersection. Therefore $\bigoplus_{\mathbf{n} \in \mathbb{Z}} (0 :_{M_{\mathbf{n}}} I)(-\mathbf{n}) = 0 :_{\mathcal{M}} I = \bigcap_{i=1}^r \ker(\varphi_i)$ is a constructible G -module. $\square$

Definition 9.26. Fix a Q -module M , any monomial prime $\mathfrak{p} \subseteq \mathbb{k}[Q_+]$, and a linear function $\lambda : Q \rightarrow \mathbb{Z}$. The v -invariant of M at $\mathfrak{p}$ in direction λ is

$$v_{\mathfrak{p}}(\lambda, M) = \min\{|\lambda, \mathbf{u}| \mid \mathfrak{p} = I :_R f \text{ for some } f \in R_{\mathbf{u}}\}.$$

The case $\lambda = (1, \dots, 1)$ is Definition 9.23.

Theorem 9.27. *For a constructible family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}}$ of Q -modules, monomial prime $\mathfrak{p}$, and linear $\lambda : Q \rightarrow \mathbb{Z}$, the v -invariants $v_{\mathfrak{p}}(\lambda, M_{\mathbf{n}})$ in direction λ are piecewise quasilinear.*

Proof. Let J be the product of the (finitely many) monomial prime ideals properly containing $\mathfrak{p}$, with $J = R$ if no such prime exists. The proof of [Con24, Lemma 1.2] applies more generally to our setting to show that for a Q -module M the set of homogeneous elements $\{f \in M \mid \mathfrak{p} = 0 :_R f\}$ are those whose images in $(0 :_M \mathfrak{p}) / (0 :_M (\mathfrak{p} + J^\infty))$ is nonzero, where $0 :_M (\mathfrak{p} + J^\infty) = \bigcup_{m \geq 0} (0 :_M (\mathfrak{p} + J^m))$. The conclusion now follows by applying Theorem 3.11.2 to the family $\{(0 :_{M_{\mathbf{p}}} \mathfrak{p}) / (0 :_{M_{\mathbf{n}}} (\mathfrak{p} + Q^\infty))\}_{\mathbf{n} \in \mathbb{Z}}$, which is constructible by Lemma 9.25. $\square$

Remark 9.28. Theorem 9.27 recovers for monomial ideals results in [FS25] and [AS25] about linear behavior of v -invariants of noetherian graded families of ideals. Moreover, our theorem shows that for monomial ideals $I_1, \dots, I_k$ and a monomial prime ideal $\mathfrak{p}$, the v -invariants $v_{\mathfrak{p}}(I_1^{n_1} \cdots I_k^{n_k})$ are piecewise quasilinear.

9.5. Degrees and multiplicities.

Definition 9.29. Recall Convention 9.1 regarding $\mathfrak{m} \subseteq R = \mathbb{k}[Q_+]$. Fix an $\mathfrak{m}$ -primary ideal I , so $(\mathbf{x}^{\mathbf{q}})^n \in I$ for $n \gg 0$ for every nonunit monomial $\mathbf{x}^{\mathbf{q}}$, and a noetherian R -module M . The *Hilbert polynomial (of M with respect to I)* is the polynomial

$$H_{I,M}(m) \in \mathbb{Q}[m] \text{ of degree } \delta = \dim(M)$$

such that $\ell(M/I^m M) = H_{I,M}(m)$ for all $m \gg 0$. When this polynomial is written as

$$H_{I,M}(m) = e_0(I, M) \binom{m + \delta}{\delta} m^{\delta} - e_1(I, M) \binom{m + \delta - 1}{\delta - 1} m^{\delta-1} + \cdots + (-1)^{\delta} e_{\delta}(I, M),$$

the integers $e_i(I, M)$ are the *Hilbert coefficients*. The coefficient $e_0(I, M) \in \mathbb{N}$ is commonly referred to as the *Hilbert-Samuel multiplicity*. When $I = \mathfrak{m}$, the leading coefficient $e_0(\mathfrak{m}, M)$ is also called the *degree of M* and denoted by $\deg(M)$.

Theorem 9.30. Fix a constructible family $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^k}$ of noetherian Q -modules and an $\mathfrak{m}$ -primary monomial ideal I in the affine semigroup ring $R = \mathbb{k}[Q_+]$. The Hilbert coefficients $e_i(I, M_{\mathbf{n}})$ for each fixed i are piecewise quasipolynomial of degree at most d .

Proof. The family $\{M_{\mathbf{n}}/I^m M_{\mathbf{n}}\}_{m\mathbf{n} \in \mathbb{Z}^{k+1}}$ is constructible by Proposition 7.29. Thanks to Theorem 3.11.1 the lengths $\ell(M_{\mathbf{n}}/I^m M_{\mathbf{n}})$ are piecewise quasipolynomial of degree at most d . Let $\mathbb{Z}^{k+1} = \bigcup_{\alpha \in A} (\Gamma_{\alpha} \cap \mathbb{Z}^{k+1})$ be a finite polyhedral partition afforded by Definition 3.8, with the piecewise quasipolynomial $\ell(M_{\mathbf{n}}/I^m M_{\mathbf{n}})$ equal to the quasipolynomial $Q_{\alpha}(m, \mathbf{n})$ on Γ_{α} . The dimensions $\dim(M_{\mathbf{n}})$ are quasiconstant by Theorem 9.4.4, so further assume that if $P(m, \mathbf{n})$ is one of the polynomials that define $Q_{\alpha}(m, \mathbf{n})$, then $\dim(M_{\mathbf{n}})$ is constant on the values of $\mathbf{n}$ for which $Q_{\alpha}(m, \mathbf{n}) = P(m, \mathbf{n})$ for $m\mathbf{n} \in \Gamma_{\alpha}$.

Let $A' \subseteq A$ be the subset indexing the polyhedra Γ_{α} in the partition such that $S_{\alpha} = \{\mathbf{n} \mid m\mathbf{n} \in \Gamma_{\alpha} \text{ for } m \gg 0\}$ is nonempty. The sets S_{α} for $\alpha \in A'$ polyhedrally partition of $\mathbb{R}^k$. Fix $\alpha \in A'$ and $\mathbf{n}_0 \in S_{\alpha}$. Then $Q_{\alpha}(m, \mathbf{n}_0)$ coincides with the Hilbert polynomial $H_{I, M_{\mathbf{n}_0}}(m)$ for $m \gg 0$. Since Q_{α} is defined by finitely many polynomials, $Q_{\alpha}(m, \mathbf{n}_0) = H_{I, M_{\mathbf{n}_0}}(m)$ for every m with $m\mathbf{n}_0 \in \Gamma_{\alpha}$. Thus, for every $\mathbf{n} \in S_{\alpha}$ the Hilbert coefficient $e_i(I, M_{\mathbf{n}})$ is the coefficient of $m^{\dim(M_{\mathbf{n}})-i}$ in $Q_{\alpha}(m, \mathbf{n})$, which is a quasipolynomial in $\mathbf{n}$ of degree at most d . $\square$

Remark 9.31. Theorem 9.30 answers in the affirmative (for constructible families) several questions in [HPV08, Introduction] on the quasipolynomial behavior of Hilbert coefficients of several sequences of modules, such as that of $e_i(I^n M/J^n N)$. Theorem 9.30 also extends several related results already included in [HPV08].

Definition 9.32 ([Vas98a, Definition 2.8]). Assume $\mathbb{k}[Q_+]$ is Gorenstein and that Q_+ is generated by elements that lie on a hyperplane (for example, $Q_+ = \mathbb{N}^d$). The *homological degree* of a noetherian $\mathbb{N}$ -graded module M of Krull dimension $\delta = \dim(M)$ is defined recursively by

$$\text{hdeg}(M) = \deg(M) + \sum_{i=d-\delta+1}^d \binom{\delta-1}{i-d+\delta-1} \text{hdeg}(\text{Ext}_R^i(M, R)).$$

Vasconcelos introduced homological degrees with the goal of extending good properties of degrees to modules that are not Cohen–Macaulay, for instance to provide bounds for numbers of generators.

Corollary 9.33. *Assume Q_+ satisfies the hypothesis in Definition 9.32. Let $\{M_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}}$ be a constructible family of noetherian Q -modules. The homological degrees $\text{hdeg}(M_{\mathbf{n}})$ are piecewise quasipolynomial of degree at most d .*

Proof. Since $\text{hdeg}(M_{\mathbf{n}})$ is a linear combination of $\deg(M_{\mathbf{n}})$ and degrees of iterated Ext modules of the form $\text{Ext}_R^{i_1}(\text{Ext}_R^{i_2}(\cdots(\text{Ext}_R^{i_r}(M_{\mathbf{n}}, R), R), \cdots, R)$, the result is a consequence of Theorem 9.30 and Theorem 8.8. $\square$

Remark 9.34. As far as we are aware, this polynomial behavior for homological degrees was not known, even for quotients $\{R/I^n\}_{n \in \mathbb{N}}$ by powers of a fixed monomial ideal.

Remark 9.35. The same statement and proof work for the *arithmetic degree*

$$\text{adeg}(M) = \sum_{\mathfrak{p} \in \text{Ass } M} \text{mult}_M(\mathfrak{p}) \deg(R/\mathfrak{p})$$

[Vas98a, Definition 2.3] by its characterization [Vas98a, Proposition 2.4] as a sum over double Ext modules. It would be difficult to verify whether arbitrary extended degrees [Vas98b, p. 347], such as *unmixed degree* [CQ25], behave piecewise quasipolynomially.

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