Symmetry: A Mathematical Exploration. By Kristopher Tapp. Springer, New York, 2012, xiv + 215pp., ISBN 978-1-4614-0298-5, $49.95.

Reviewed by **Kevin Woods**

“Why are honeycombs hexagonal?” “Why did the HIV virus evolve its icosahedral shape?” “How might the symmetry in a painting enhance its artistic appeal?” Kristopher Tapp asks these questions in his preface, and assures us that this book is a good place to start if we want to answer them; understanding symmetry groups and classifying possible symmetry types is a “crucial prerequisite for addressing questions” such as these.

This book is designed for a Mathematics for Non-Majors style class: a terminal course with no college-level prerequisites, which will attract students from a broad variety of mathematical backgrounds and interest-levels. I have not taught a course of this style, but I have enjoyed teaching a First Year Seminar similarly dedicated to symmetry. We read about symmetry in the artistic and natural worlds, we learn a little group theory, and I promise them that the mathematics will provide insights into questions like the ones Tapp asks in the preface.

By the end of the semester, I am crossing my fingers that the students have forgotten my promise. Sure, group theory plays a crucial role in understanding modern physics, but the math gets too hard too quickly for my students, so we rely on metaphors rather than Lie Algebras. Sure, one can classify designs from various cultures as one of the 17 wallpaper symmetry groups (see [7]), but understanding how that should inform a discussion on aesthetics is difficult. Sure, we discuss chirality and its importance in molecules and biology, but the mathematics is no deeper than understanding the difference between your left and right hands.

Upon reading Tapp’s preface, I was crossing my fingers that he had fixed this hole in my class. It is an excellent book, and I recommend it for someone teaching a non-majors course on symmetry, but it does fall short in this respect. Let me first tell you about my favorite chapter in the book, which *is* successful at finding this synergy between the mathematics of symmetry and the natural world.

1 The Contest

Tapp draws us into Chapter 9 with a story: "All the farmers in the land competed in a contest to design the least-perimeter fence enclosing a given area." Drama! We will be proving the Isoperimetric Inequality, that the circle should be the contest winner. "Farmer Don won! His fence not only beat the other farmers' fences but it also beat all possible other fences." (Relax! Tapp soon comes clean that there is a major plot hole here.) I'll condense the rest of the proof:

Consider a vertical line dividing Don's fenced-in area into two pieces, L and R, of equal area. Now examine two possible contenders in the contest: the first is L together with its mirror image L', and the second is R with its mirror image R'. These enclose the same area as Don's fence, and since we are told that neither beat his fence by having a shorter length, they must all be tied. Therefore we can assume that the prize winning fence has a vertical symmetry. Furthermore, the fence must meet this reflection axis perpendicularly: if it didn't, either $L \cup L'$ or $R \cup R'$ would have an inward "notch" at this axis, and we could modify the fence to make it shorter while enclosing more area. Repeating the process with a horizontal mirror gives a winning fence with the symmetry of a rectangle, including 180 degree rotational symmetry. Now take any line through the center of rotation. The rotational symmetry implies that the line divides the area in half. The same mirroring trick then shows that the line must meet the fence at right angles.

Since this is true for any line through the center, the only shape the fence could have is a circle. In order to avoid calculus, Tapp can only sketch a proof of this: we imagine tracing the fence with a compass that we can open or close as needed to stay on the fence. Opening or closing the compass, however, yields obtuse or acute angles between the fence and the radial line, so we must keep the compass rigid, and we end up drawing a perfect circle. Tapp admits that a full proof of this would require calculus, but hopes this sketch is convincing. I'm convinced, and I think the students will be too.

The logical structure of the proof is nontrivial; this is real math, and it's helping us answer a real question. Soap bubbles are spheres – why? Answer: if they had less symmetry than a sphere does, we would be able to exploit that asymmetry to build a better bubble.

2 Engaging with Mathematics

I believe that a primary goal of a non-majors class should be to engage the students. That's almost an empty statement; it should be a goal of any class to engage the students, as engagement is a prerequisite to learning, but accomplishing this can be more challenging in a non-majors course. Some students (alas) will not be excited about math for its own sake, but ideas like the fence contest provide a hook: we see that mathematics helps us understand interesting things, and hopefully we transition to realizing that the mathematics itself is interesting.

It also helps that Tapp's enthusiasm bleeds from the page; while I have never met him and don't know what he sounds like, I can hear the excitement in his voice as the story captures me. What I find most admirable is how rigorous Tapp makes the proof, while keeping it engaging, a task which could be difficult given the intended audience. When there are important holes in an argument, Tapp admits it, as he does here when he confesses that the hardest part of the proof may be to show that there exists a winning fence at all.

As important as engaging the students is, we want them engaged in the right thing. What is that? In my First Year Seminar, I have the luxury of the right thing being pretty much anything that gets them thinking critically. For example, we discuss E.H. Gombrich's *The Sense of Order* [4], which seeks to explain the aesthetics of repetitive symmetry patterns using the psychology of perception. There is little mathematics here, which is acceptable for my class, but unacceptable for a Math for Non-Majors course; students should be engaged in mathematics, indeed *interesting* mathematics. This fence contest chapter does that perfectly. We assume that something is minimal and explore the implications. We learn things that are true without loss of generality (that Farmer Don's winning fence has rectangular symmetry). We prove by contradiction that the fence meets the radial lines at right angles. We use local properties (that the compass can neither open nor close) to prove global properties (the fence is a circle).

One more amendment to my desired outcome in this class: I want the students *actively* engaged in interesting mathematics. I want students to see mathematics as a process that they can participate in. Ironically, Tapp's proof is so lucid that it leaves little room for exploration, only admiration. The exercises would be a good place for students to learn to explore, and there are some good ones in this chapter. In general, I wish there were more (this section has six short exercises, a mix of basic and challenging problems). In particular, I'd like to see more in-depth, guided exercises that help students understand the material and discover new ideas.

For example, Tapp has a good, basic question about a related optimization problem: we want to figure out the shortest network of roads that can connect four cities, situated at the vertices of a square; students must compute total road lengths for an "X", an "H", and a "U" network. How about an actual contest among the students? Start with only *three* points, on a 20×20 grid, and challenge students to connect them with as short a road network as possible. Repeat with several different configurations. They will discover for themselves an exciting and non-trivial fact: they should send a road out from each point, and the three roads should meet at a center point, making three 120 degree angles at the intersection. After students have themselves discovered it for three points, the solution for four points on a square (the network should be somewhere between an "H" and an"X", again with 120 degree angles all around) will be more motivated and interesting. And now we're ready to talk about symmetry *breaking*: the solution has less symmetry that the square, something Tapp had previously shown the reader with soap bubbles.

3 The Rest of the Book

I picked my favorite chapter to discuss in depth, which comes right after the heart of the book, the first eight chapters that build up the theory of symmetry groups. In these, we develop some basic groups: cyclic, dihedral, translation, and permutation groups (including alternating groups). We learn about group isomorphisms and classify two-dimensional finite and wallpaper groups. Finally, we classify three-dimensional finite groups, culminating in understanding the symmetry groups of the Platonic solids.

This book doesn't shy away from challenging proofs and concepts. It is amazing to think how much students will have learned by the time they, for example, can conceptualize the rotational symmetry group of the cube as the permutation group acting on the four diagonals. Any time a proof can be explained reasonably, it is (lucidly!). This includes many mathematically interesting ideas, like using conjugation to understand why two congruent shapes have isomorphic symmetry groups. Any time a proof has been left out, I've looked one up and realized that it either involves too many new concepts (showing that all permutations are either even or odd) or is too tedious (classifying all wallpaper groups).

Chapters 10 and 11 introduce the real and rational numbers, leading up to proving the uncountability of the reals. This is certainly beautiful material. It is not, however, particularly tied to symmetry. If you want to cover this material in your class, it is as well-presented as the rest of the book. If you don't want to cover it, you're in a bit of a bind: the book is fairly short, so there isn't much room to pick and choose in a semester course.

In the early chapters, Tapp admits that for a truly rigorous understanding of symmetries and rigid motions, we'll need to understand matrices. He holds off until Chapter 12 to teach us this material. I'm surprised by the delay; most students will have previously been exposed to matrices, trig, and Euclidean space, and it seems like this material could be integrated into the first half of the book.

4 Other Books

I have seen a couple other excellent books that could be used in a non-majors course on symmetry. The first is *Groups and Symmetry: A Guide to Discovering Mathematics*, by David Farmer [3]. The text of this book expects less of the reader than Tapp's; there are not as many hard theorems, concepts, and proofs. It does discuss cyclic, dihedral, and permutation groups, but it leaves out isomorphisms and three-dimensional groups. The exercises of this book, on the other hand, expect *more* of the reader than Tapp's book. In fact, this book is really built on the exercises, which are interspersed among definitions and discussion.

Here is an example of a challenging exercise in Farmer's book: After seeing many examples of symmetries of a horizontal, one-dimensional strip, students are led through a classification of all possible symmetry types. Previously in the chapter, the key fundamental symmetries (horizontal/vertical reflections, 180 degree rotation, and glide reflection) have been seen, and six of the seven possible symmetry types have been shown – I think ... this is not easy stuff to check! Students are guided to do two things simultaneously: list all of the strip patterns they can, together with their fundamental symmetries, and list rules for what combinations of fundamental symmetries are allowed (a strip with a horizontal reflection always has a glide reflection; a strip with horizontal and vertical reflections always has a 180 degree rotation).

Once these two lists are complete, we will see that we've come up with all possibilities.

Clearly this is a hard problem. I expect it would take the entire class, working together and with instructor guidance, to get it. But that's part of the point, seeing mathematics as a process of discovery, conjecturing, and testing. There is also less rigor here than in Tapp's book; for example, the hardest thing to check is what has been swept under the rug: two strips with, say, only glide reflections and vertical reflections must have the same symmetry "type". What "type" even means is left unsaid in Farmer's book, but it is explicit in Tapp's: exactly the same Euclidean symmetry group after translation and rescaling. I am willing to give up a little rigor for the joy of discovery; my ideal textbook lies between Tapp's and Farmer's.

Another interesting book is *Symmetry, Shape, and Space: An Introduction to Mathematics Through Geometry*, by Kinsey and Moore [5]. This book expects less of the reader in both the text and the exercises. In contrast to Tapp's and Farmer's, this one is long, containing way more than enough material for a semester. It covers symmetry in two and three dimensions, adding chapters on tessellations and spirals that the other books lack. It surveys many other topics, as well: constructions, curvature, graph theory, and topology. It does not include any group theory, and generally is light on theorems and proofs. It does, however, contain many great exercises, integrated into the sections, most with a geometric and hands-on emphasis. This is a fun book that will engage students; depending on the student, it may not push them enough.

And I have to mention *The Symmetries of Things*, by Conway, Burgiel, and Goodman-Strauss [1]. This book is too challenging for a non-majors course, but beautifully illustrated and written. Send students to it who want enrichment beyond what the other books can offer, and read it yourself!

5 Conclusion

The Fence Contest was my favorite part of Tapp's book. Interesting and approachable mathematics of symmetry is used to prove something insightful about the world. Not only does this context help draw the student in to engage actively in the mathematics, but it helps them leave the course feeling that the mathematical reasoning they have learned is important in their life. There aren't many more results like that in the book; the next best example that I can think of is a proof that there are too many dimples on a golf ball for them to be spaced out symmetrically.

This is unfortunate, I think. As Lynn Steen states [6, p:47] in his book reviewing recent trends in quantitative literacy at the college level, "Almost without exception, everyone who engages the issue of quantitative literacy concludes that 'in context' is one of its defining features." One of our goals in a non-majors class is to help students reason formally and carefully, in the hopes that they will be able to think more critically in their everyday lives. We want students to be able to apply these reasoning skills in unfamiliar contexts, which suggests we need to teach these skills in context, in the first place. Otherwise, we risk them remembering symmetry groups as a fun game they played, one time in college. "For students, context creates meaning" [6, p:24].

I can't fault Tapp's book for this deficiency, because I have also failed in finding similar material for my First Year Seminar on symmetry. If I were teaching a non-majors class on symmetry groups, I would use this book (supplemented by exercises from Farmer's). I'm not sure that I want to teach that class though. I love the abstractness of group theory. Will my students? Maybe I'm selling them short, but there are books on other subjects that inculcate mathematical reasoning just as well, while remaining more closely tied to the real world. A textbook of the sort that I am excited about right now is *Networks, Crowds, and Markets: Reasoning about a Highly Connected World*, by Easley and Kleinberg [2], which uses graph theory to understand social networks. That is a class I would prefer to teach.

Tapp's pitch-perfect voice manages to be engaging on a page-by-page basis, which is tremendously important. I think the hope is that the inherent concreteness of geometry – building dodecahedra, drawing a wallpaper pattern, cutting out a triangle and flipping it over – will provide the necessary broad engagement in the material. In the end, the instructor's enthusiasm is crucial; I'm sure Tapp's excitement, so evident on the page, is equally intense when he teaches the class, and if you are excited by teaching this material, this book is a great choice.

I've yet to mention the first thing you will notice when you open the book: the appearance. The font is large, the margins are small, and there is colored text and graphics everywhere. Initially this was off-putting to me, because it is different from almost any other math textbook. And I suspect that tension is the rationale for the choice; it communicates to the student that this is unlike the math textbooks they had in high school. Certainly, I quickly grew accustomed to it.

In conclusion, this is a textbook that doesn't pull its punches. It is challenging and rigorous, while being approachable. It would be a perfect independent study for a young student who is excited about and somewhat adept at math. It could be a great choice as a text for a class, as long as the instructor is willing to work hard at helping the students get many of the abstract details and is willing to supplement the exercises with their own.

References

[1] J.H. Conway, H. Burgiel, C. Goodman-Strauss, *The Symmetries of Things*, A K Peters, Wellesley, Massachusetts (2008).

[2] D. Easley, J. Kleinber, *Networks, Crowds, and Markets: Reasoning about a Highly Connected World*, Cambridge University Press, New York (2010).

[3] D.W. Farmer, *Groups and Symmetry: A Guide to Discovering Mathematics*, American Mathematical Society, Providence (1996).

[4] E.H. Gombrich, *The Sense of Order: A Study in the Psychology of Decorative Art*, The Phaidon Press, London (1979).

[5] L.C. Kinsey, T.E. Moore, *Symmetry, Shape, and Space: An Introduction to Mathematics Through Geometry*, Key College Publishing, Emeryville, California (2002).

[6] L.A. Steen, *Achieving Quantitative Literacy: An Urgent Challenge for Higher Education*, Mathematical Association of America, Washington, DC (2004).

[7] D.K. Washburn, D.W. Crowe, *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*, University of Washington Press, Seattle (1988).

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