Quasi-polynomial Behavior in Factorizations via Presburger Arithmetic

Kevin Woods
Oberlin College

Some Nice Functions, $S = \langle 2, 3, 6 \rangle$

Let $S = \langle 2, 3, 6 \rangle = \{0, 2, 3, 4, 5, 6, \ldots\}$, the additive semigroup (monoid) generated by 2, 3, and 6.

- Boring, and the 6 is redundant.

Definition: Given $a \in S$, a factorization of a is $(x, y, z) \in \mathbb{N}^3$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$) such that $2x + 3y + 6z = a$.

- A factorization demonstrates that $a \in S$.

Example: 8 has three factorizations: (4,0,0), (1,2,0), and (1,0,1).

- A little more interesting now, at least.

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$$f(t) = \begin{cases} \binom{\lfloor t/6 \rfloor + 2}{2}, & \text{if } t \equiv 0, 2, 3, 4, 5 \pmod{6}, \\ \binom{\lfloor t/6 \rfloor + 1}{2}, & \text{if } t \equiv 1 \pmod{6}. \end{cases}$$

- Can you figure out why? Hint: factorizations will (approximately) break up t into $\approx t/6$ groups of 6 (three 2's, two 3's, or one 6).

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Another way of writing it:

$$f(t) = \begin{cases} \frac{1}{72}t^2 + \frac{1}{4}t + 1, & \text{if } t \equiv 0 \pmod 6, \\ \frac{1}{72}t^2 + \frac{1}{18}t - \frac{5}{72}, & \text{if } t \equiv 1 \pmod 6, \\ \frac{1}{72}t^2 + \frac{7}{36}t + \frac{5}{9}, & \text{if } t \equiv 2 \pmod 6, \\ \frac{1}{72}t^2 + \frac{1}{6}t + \frac{3}{8}, & \text{if } t \equiv 3 \pmod 6, \\ \frac{1}{72}t^2 + \frac{5}{36}t + \frac{2}{9}, & \text{if } t \equiv 4 \pmod 6, \\ \frac{1}{72}t^2 + \frac{1}{9}t + \frac{7}{72}, & \text{if } t \equiv 5 \pmod 6. \end{cases}$$

Such a formula is called a quasi-polynomial: there exist a period m and polynomials $f_0, f_1, \ldots, f_{m-1}$ such that

$$f(t) = f_{t \bmod m}(t), \forall t \in \mathbb{N}.$$

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Now let $f(t)$ be the number of distinct factorization lengths of t. So $f(8) = 3$. Does this f have a nice formula?

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Yes!

$$f(t) = \begin{cases} \left\lfloor \frac{t}{2} \right\rfloor - \left\lceil \frac{t}{6} \right\rceil + 1, & \text{if } t \equiv 0, 2, 3 \pmod 6, \\ \left\lfloor \frac{t}{2} \right\rfloor - \left\lceil \frac{t}{6} \right\rceil, & \text{if } t \equiv 1, 4, 5 \pmod 6,\ t > 1, \\ 0, & \text{if } t = 1. \end{cases}$$

- Lengths must be between $\left\lceil \frac{t}{6} \right\rceil$ (all 6's) and $\left\lfloor \frac{t}{2} \right\rfloor$ (all 2's), so all possible lengths (or all but one) are realized.

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Let $f(s, t)$ be the number of factorization of t of length s.

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where $c_{s,t} = (\lfloor t/6 \rfloor + s \bmod 2)$.

This is a piecewise quasi-polynomial (where the pieces are defined by linear inequalities in the parameters).

- The last piece has period 2 in s and period 12 in t.

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$$f(r, s, t) = \begin{cases} 2r + 3s + 6t - 1, & \text{if } s \geq 1, r \geq 2, \\ (2r + 3s + 6t + 2)/2, & \text{if } s = 0, r \geq 2, \\ \text{etc.} \end{cases}$$

- The semigroup element must be between 0 and $2r + 3s + 6t$, so this is all but two possibilities.
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Presburger Arithmetic

What do all of these combinatorial questions have in common?

They count the number of elements in a set defined in Presburger arithmetic: sets defined over the integers using:

- linear (in)equalities,
- boolean operations ($\wedge$, $\vee$, $\neg$), and
- quantifiers ($\exists$, $\forall$).

Factorizations of t:

$$(x, y, z) \in \mathbb{N} : \; 2x + 3y + 6z = t.$$

Lengths of factorizations of t:

$$\ell \in \mathbb{N} : \; \exists x, y, z \in \mathbb{N}, \; (2x + 3y + 6z = t) \; \wedge \; (x + y + z = \ell).$$

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$$a \in \mathbb{N} : \ \exists x, y, z \in \mathbb{N}, \ (2x+3y+6z = a) \wedge (x \leq r) \wedge (y \leq s) \wedge (z \leq t).$$

Presburger Arithmetic

Theorem [W]: For any sets defined in Presburger arithmetic, the corresponding counting functions will be piecewise quasi-polynomials.

Note: There are three types of variables (parameters, counted, quantified) that may appear in the linear inequalities:

$$f(t) = \#\ell \in \mathbb{N} : \exists x, y, z \in \mathbb{N}, \ (2x+3y+6z = t) \wedge (x+y+z = \ell).$$

Note: For one parameter variable, “piecewise” quasi-polynomial is equivalent to “eventually” a quasi-polynomial.

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Extensions

The theorem is actually more general.

The largest/smallest element of a set can be picked out, and it will have a piecewise/eventually quasi-polynomial structure:

Example: The lexicographically largest factorization of t of length s is

$$\left(\frac{6s-t-9}{4},\ 3,\ \frac{t-2s-3}{4}\right),\ \text{if } t \equiv 3 \pmod 4,\ s \equiv 0 \pmod 2,$$

and so forth.

Also, the parameters for which a Presburger sentence is true will be piecewise/eventually periodic:

Example: $t \in \langle 6, 8 \rangle$ if and only if $t \equiv 0 \pmod 2$, for $t > 10$.

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Everything I'm saying here also applies to affine semigroups (additive semigroups in $\mathbb{Z}^k$) as well.

- All of the constraints are still linear, so expressible in Presburger arithmetic.

Example: For the affine semigroup $\langle\, (1,2),(2,2),(3,2)\,\rangle$, the number of factorizations of (s,t) is

$$\frac{2t-s+2}{2},\ \text{if}\ \frac{3}{2}t \le s \le 2t,\ s \equiv 0 \pmod 2,$$

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A Twist

So far we've fixed the semigroup, but what if we let it vary with t?

Example: $S_t = \langle t, t+1, t+3 \rangle$. (in general, we could have S_t generated by any polynomials in t).

Then S_t can be defined by

$$a \in \mathbb{N} : \exists x, y, z \in \mathbb{N},\ \ tx + (t+1)y + (t+3)z = a.$$

These are nonlinear in t!

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And yet we still get nice functions!

Let $f(t)$ be the number of gaps of S_t (positive integers not in S_t).

$$f(t) = \begin{cases} \frac{1}{6}t^2 + \frac{1}{2}t, & \text{if } t \equiv 0 \pmod 3, \\ \frac{1}{6}t^2 + \frac{1}{2}t - \frac{2}{3}, & \text{if } t \equiv 1 \pmod 3, \\ \frac{1}{6}t^2 + \frac{1}{2}t - \frac{2}{3} & \text{if } t \equiv 2 \pmod 3. \end{cases}$$

Let $f(t)$ be the Frobenius number of S_t (the largest integer not in S_t).

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This works in the following setting, called **parametric** Presburger arithmetic:

- Require a **single** parameter, t.
- Allow multiplication by this parameter (but not by other variables).
- So base inequalities are of the form

  $$p_1(t)x_1 + \cdots + p_n(t)x_n \leq q(t),$$

  where $p_i, q \in \mathbb{Z}[t]$. For fixed t, these are just linear inequalities.
- Still allow Boolean operations and quantifiers.

Then you still get eventual quasi-polynomials!
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Almost any invariant or property of a semigroup can now be applied to S_t as a function of t. As long as is can be defined in parametric Presburger arithmetic, you will get eventual quasi-polynomial behavior.

We give many examples in [Bogart–Goodrick–W].

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Given a family of semigroups, S_t, generated by polynomials in t, the following (when finite) are eventually quasi-polynomial:

- The number of gaps,
- The Frobenius number,
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- The number of fundamental gaps,
- The cardinality of the delta set,
- For fixed i, the value of the ith largest element of the Apéry set

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Example: The Frobenius number, a, is defined by

$$a \notin S_t \;\wedge\; \forall b > a,\; b \in S_t,$$

can be expanded out to a parametric Presburger sentence.

Example: The set of pseudo-Frobenius numbers is

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Given a family of semigroups, S_t, generated by polynomials in t, the set of t such that S_t has the following properties is eventually periodic:

- S_t is a numerical semigroup.
- S_t is a symmetric numerical semigroup
- S_t is an irreducible numerical semigroup.

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Not Always So Easy

The examples on the previous slides tended to be pretty easy to prove definable in parametric Presburger arithmetic. Some are harder:

Theorem (Bogart–Goodrick–W)

For every field k, the ith Betti number of the semigroup algebra $k[S_t]$ is eventually quasi-polynomial.

Key: Try to find some finite structure (independent of t) that can be encoded in parametric Presburger arithmetic.

- Look at the fixed number of simplicial complexes on the generators. Do a homology calculation on those that have a certain property (squarefree divisor complex for S_t) that is definable in parametric Presburger [Bruns–Herzog].

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Thank You!