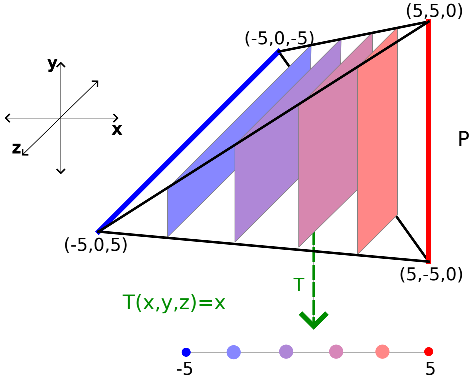
Solving Lattice Point Problems Using Rational Generating Functions

Kevin Woods
Oberlin College



An Easy Start

Question: How many even numbers are there between 100 and 250?

An Easy Start

Question: How many even numbers are there between 100 and 250?

List them all:

100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128,

130, 132, 134, 136, 138, 140, 142, 144, 146, 148, 150, 152, 154, 156, 158,

160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188,

190, 292, 294, 296, 298, 200, 202, 204, 206, 208, 210, 212, 214, 216, 218,

220, 222, 224, 226, 228, 230, 232, 234, 236, 238, 240, 242, 244, 246, 248,

250

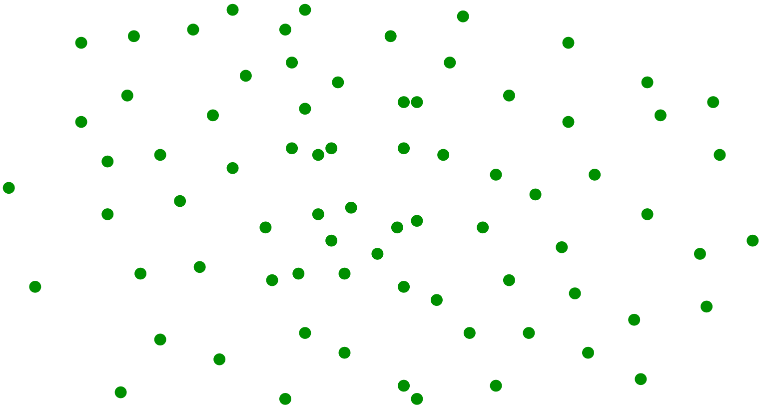
and count: 76.

An Easy Start

This is the wrong way to answer the question.

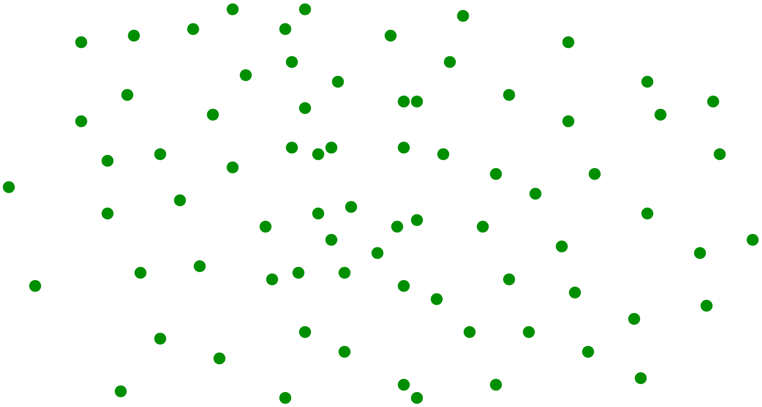
Another Easy One

Question: How many dots are in this picture?



Another Easy One

Question: How many dots are in this picture?



Count them: 76.

This is the best we can do.

Philosophy Class

The difference:

Philosophy Class

The difference:

The set of even numbers between 100 and 250 has a pattern that we can take advantage of.

Theme of talk: Demonstrate a nice tool to take advantage of the special structure of certain sets.

That tool is generating functions.

The Easy Problem, Redux

Given a set $S \subseteq \mathbb{N}$, define the generating function

$$f(S;x) = \sum_{a \in S} x^a.$$

In example,

$$\begin{aligned} f(S;x) &= x^{100} + x^{102} + x^{104} + \cdots + x^{248} + x^{250} \\ &= \frac{x^{100} - x^{252}}{1 - x^2}. \end{aligned}$$

Then $|S| = f(S;1)$.

Use l'Hospital's rule:

$$f(S;1) = \frac{100 - 252}{-2} = 76.$$

The Easy Problem, Redux

Given a set $S \subseteq \mathbb{N}$, define the generating function

$$f(S;x) = \sum_{a \in S} x^a.$$

In example,

$$\begin{aligned} f(S;x) &= x^{100} + x^{102} + x^{104} + \cdots + x^{248} + x^{250} \\ &= \frac{x^{100} - x^{252}}{1 - x^2}. \end{aligned}$$

Then $|S| = f(S;1)$.

Use l'Hospital's rule:

$$f(S;1) = \frac{100 - 252}{-2} = 76.$$

The Easy Problem, Redux

Given a set $S \subseteq \mathbb{N}$, define the generating function

$$f(S;x) = \sum_{a \in S} x^a.$$

In example,

$$\begin{aligned} f(S;x) &= x^{100} + x^{102} + x^{104} + \cdots + x^{248} + x^{250} \\ &= \frac{x^{100} - x^{252}}{1 - x^2}. \end{aligned}$$

Then $|S| = f(S;1)$.

Use l'Hospital's rule:

$$f(S;1) = \frac{100 - 252}{-2} = 76.$$

The Easy Problem, Redux

Given a set $S \subseteq \mathbb{N}$, define the generating function

$$f(S;x) = \sum_{a \in S} x^a.$$

In example,

$$\begin{aligned} f(S;x) &= x^{100} + x^{102} + x^{104} + \cdots + x^{248} + x^{250} \\ &= \frac{x^{100} - x^{252}}{1 - x^2}. \end{aligned}$$

Then $|S| = f(S;1)$.

Use l'Hospital's rule:

$$f(S;1) = \frac{100 - 252}{-2} = 76.$$

The Frobenius Problem

Let $a_1, a_2, \ldots, a_d$ be nonnegative integers such that $\gcd(a_1, a_2, \ldots, a_d) = 1$. Let

$$S = \{\lambda_1 a_1 + \cdots + \lambda_d a_d : \ \lambda_i \in \mathbb{N}\}.$$

Question: What is the largest integer not in S?

Question: How many positive integers are not in S?

The Frobenius Problem

Example: $a_1 = 3$, $a_2 = 7$.

$$S = \{0, 3, 6, 7, 9, 10, 12, 13, 14, \ldots\}.$$

Question: What is the largest integer not in S?
Answer: 11.

Question: How many positive integers are not in S?
Answer: 6.

Generating Functions to the Rescue

Listing out the set is the "wrong" way to answer these questions, because there's some structure we're not using.

Let's use generating functions.

$$f(S;x) = x^0 + x^3 + x^6 + x^7 + x^9 + x^{10} + \cdots$$

As before, this can be rewritten as a nice rational function.

We will later show that

$$f(S;x) = \frac{1 - x^{a_1 a_2}}{(1 - x^{a_1})(1 - x^{a_2})}.$$

Generating Functions to the Rescue

Listing out the set is the "wrong" way to answer these questions, because there's some structure we're not using.

Let's use generating functions.

$$f(S;x) = x^{0} + x^{3} + x^{6} + x^{7} + x^{9} + x^{10} + \cdots$$

As before, this can be rewritten as a nice rational function.

We will later show that

$$f(S;x) = \frac{1 - x^{a_1 a_2}}{(1 - x^{a_1})(1 - x^{a_2})}.$$

Generating Functions to the Rescue

Listing out the set is the "wrong" way to answer these questions, because there's some structure we're not using.

Let's use generating functions.

$$f(S;x) = x^0 + x^3 + x^6 + x^7 + x^9 + x^{10} + \cdots$$

As before, this can be rewritten as a nice rational function.

We will later show that

$$f(S;x) = \frac{1 - x^{a_1 a_2}}{(1 - x^{a_1})(1 - x^{a_2})}.$$

Generating Functions to the Rescue

Let $T = \mathbb{N} \setminus S$ (which is $\{1, 2, 4, 5, 8, 11\}$ in the example).

$$f(T; x) = \frac{1}{1-x} - f(S; x)$$
$$= \frac{(1-x^{a_1})(1-x^{a_2}) - (1-x)(1-x^{a_1 a_2})}{(1-x)(1-x^{a_1})(1-x^{a_2})}.$$

The largest integer not in S is the degree of the polynomial $f(T; x)$, which is

$$(1 + a_1 a_2) - (1 + a_1 + a_2) = a_1 a_2 - a_1 - a_2.$$

The number of positive integers not in S is $f(T; 1)$, which is (taking the limit as $x \to 1$)

$$\frac{a_1 a_2 - a_1 - a_2 + 1}{2}.$$

Generating Functions to the Rescue

Let $T = \mathbb{N} \setminus S$ (which is $\{1, 2, 4, 5, 8, 11\}$ in the example).

$$\begin{aligned} f(T; x) &= \frac{1}{1-x} - f(S; x) \\ &= \frac{(1-x^{a_1})(1-x^{a_2}) - (1-x)(1-x^{a_1 a_2})}{(1-x)(1-x^{a_1})(1-x^{a_2})}. \end{aligned}$$

The largest integer not in S is the degree of the polynomial $f(T; x)$, which is

$$(1 + a_1 a_2) - (1 + a_1 + a_2) = a_1 a_2 - a_1 - a_2.$$

The number of positive integers not in S is $f(T; 1)$, which is (taking the limit as $x \to 1$)

$$\frac{a_1 a_2 - a_1 - a_2 + 1}{2}.$$

Generating Functions to the Rescue

Let $T = \mathbb{N} \setminus S$ (which is $\{1, 2, 4, 5, 8, 11\}$ in the example).

$$f(T; x) = \frac{1}{1-x} - f(S; x) = \frac{(1-x^{a_1})(1-x^{a_2}) - (1-x)(1-x^{a_1 a_2})}{(1-x)(1-x^{a_1})(1-x^{a_2})}.$$

The largest integer not in S is the degree of the polynomial $f(T; x)$, which is

$$(1 + a_1 a_2) - (1 + a_1 + a_2) = a_1 a_2 - a_1 - a_2.$$

The number of positive integers not in S is $f(T; 1)$, which is (taking the limit as $x \to 1$)

$$\frac{a_1 a_2 - a_1 - a_2 + 1}{2}.$$

Generating Functions to the Rescue

Let $T = \mathbb{N} \setminus S$ (which is $\{1, 2, 4, 5, 8, 11\}$ in the example).

$$\begin{aligned} f(T;x) &= \frac{1}{1-x} - f(S;x) \\ &= \frac{(1-x^{a_1})(1-x^{a_2}) - (1-x)(1-x^{a_1 a_2})}{(1-x)(1-x^{a_1})(1-x^{a_2})}. \end{aligned}$$

The largest integer not in S is the degree of the polynomial $f(T;x)$, which is

$$(1 + a_1 a_2) - (1 + a_1 + a_2) = a_1 a_2 - a_1 - a_2.$$

The number of positive integers not in S is $f(T;1)$, which is (taking the limit as $x \to 1$)

$$\frac{a_1 a_2 - a_1 - a_2 + 1}{2}.$$

Generating Functions to the Rescue

Let $T = \mathbb{N} \setminus S$ (which is $\{1, 2, 4, 5, 8, 11\}$ in the example).

$$f(T; x) = \frac{1}{1-x} - f(S; x)$$
$$= \frac{(1-x^{a_1})(1-x^{a_2}) - (1-x)(1-x^{a_1 a_2})}{(1-x)(1-x^{a_1})(1-x^{a_2})}.$$

The largest integer not in S is the degree of the polynomial $f(T; x)$, which is

$$(1 + a_1 a_2) - (1 + a_1 + a_2) = a_1 a_2 - a_1 - a_2.$$

The number of positive integers not in S is $f(T; 1)$, which is (taking the limit as $x \to 1$)

$$\frac{a_1 a_2 - a_1 - a_2 + 1}{2}.$$

Generating Functions to the Rescue

Let $T = \mathbb{N} \setminus S$ (which is $\{1, 2, 4, 5, 8, 11\}$ in the example).

$$\begin{aligned} f(T;x) &= \frac{1}{1-x} - f(S;x) \\ &= \frac{(1-x^{a_1})(1-x^{a_2}) - (1-x)(1-x^{a_1 a_2})}{(1-x)(1-x^{a_1})(1-x^{a_2})}. \end{aligned}$$

The largest integer not in S is the degree of the polynomial $f(T;x)$, which is

$$(1 + a_1 a_2) - (1 + a_1 + a_2) = a_1 a_2 - a_1 - a_2.$$

The number of positive integers not in S is $f(T;1)$, which is (taking the limit as $x \to 1$)

$$\frac{a_1 a_2 - a_1 - a_2 + 1}{2}.$$

What else?

Questions:

- What types of sets can be encoded as rational generating functions?
- What types of sets can be encoded as short rational generating functions, quickly?

If $S \subseteq \mathbb{N}^n$, then let

$$f(S;\mathbf{x}) = \sum_{s=(s_1,\ldots,s_n)\in S} x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}.$$

What else?

Question: When can a set be encoded as a rational generating function?

Answer [W]: If and only if it can be written like

$$S = \{x \in \mathbb{N} \;\middle|\; \forall y_1 \in \mathbb{N}, \exists y_2 \in \mathbb{N} : (3y_1 + 5y_2 - x \geq 0) \text{ and } (5y_1 + 2y_2 + 3x < 5 \text{ or } 3y_1 - x = 7)\},$$

using quantifiers ($\exists$ and $\forall$), boolean operations (and, or, not), and linear (in)equalities ($\leq$, $=$, $>$).

These are sentences in the Presburger arithmetic.

What else?

Examples:

$$S = \{x \in \mathbb{N} \;\Big|\; \exists y \in \mathbb{N} : \; 2y = x \text{ and } 100 \leq x \leq 250\}.$$

$$S = \{x \in \mathbb{N} \;\Big|\; \exists \lambda_1 \in \mathbb{N}, \ldots, \exists \lambda_d \in \mathbb{N} : x = a_1\lambda_1 + \cdots + a_d\lambda_d\}.$$

A Computer Example

```
for i=0 to 5
  for j=0 to i
    Do something that requires i · j units of storage
  end
end
```

Want to compute

$$\sum_{i=0}^{5} \sum_{j=0}^{i} ij.$$

Let

$$S = \{(i,j) \in \mathbb{N}^2 \mid i \leq 5 \text{ and } j \leq i\}.$$

We want

$$\sum_{(i,j) \in S} ij.$$

A Computer Example

```
for i=0 to 5
  for j=0 to i
    Do something that requires i · j units of storage
  end
end
```

Want to compute

$$\sum_{i=0}^{5} \sum_{j=0}^{i} ij.$$

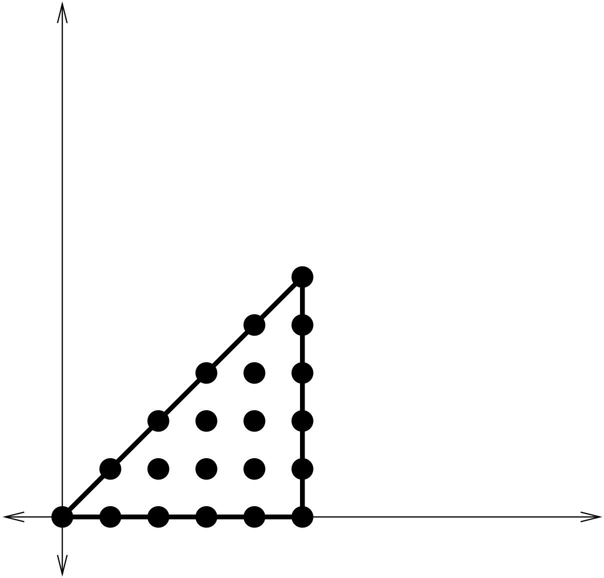
Let

$$S = \{(i,j) \in \mathbb{N}^2 \mid i \le 5 \text{ and } j \le i\}.$$

We want

$$\sum_{(i,j)\in S} ij.$$

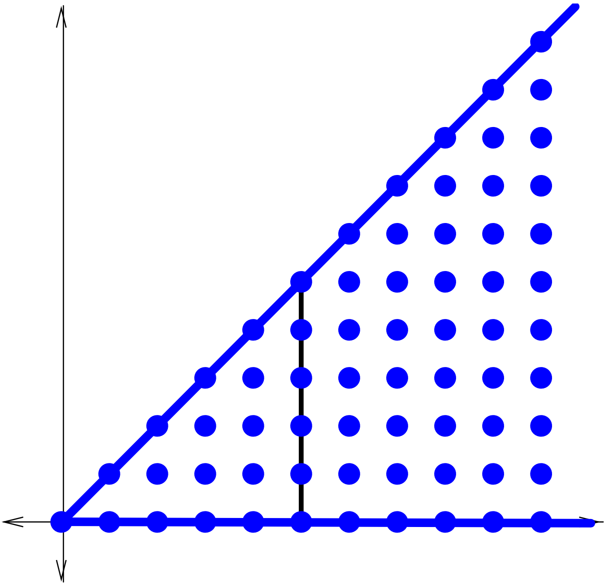
A Computer Example



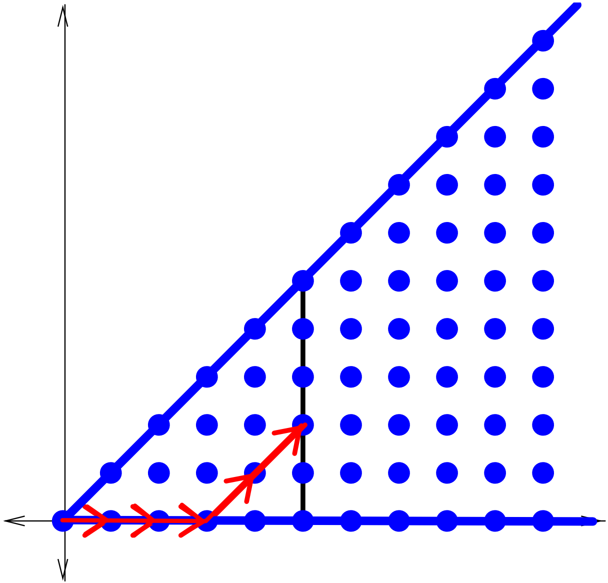
Let's find $f(S; x, y)$

$= 1 + x + \cdots + x^5 y^5$

A Computer Example



A Computer Example



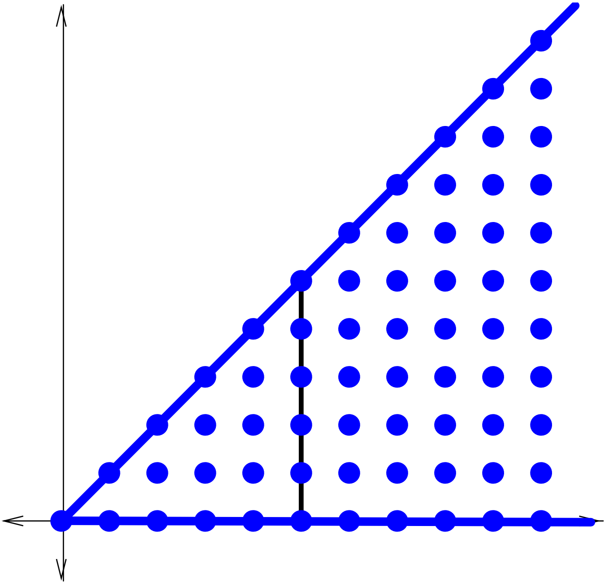
$$x^5y^2 = (x)^3(xy)^2$$

$$(1+x+x^2+x^3+\cdots)$$

$$\cdot(1+xy+(xy)^2+\cdots)$$

$$= \frac{1}{(1-x)(1-xy)}$$

A Computer Example

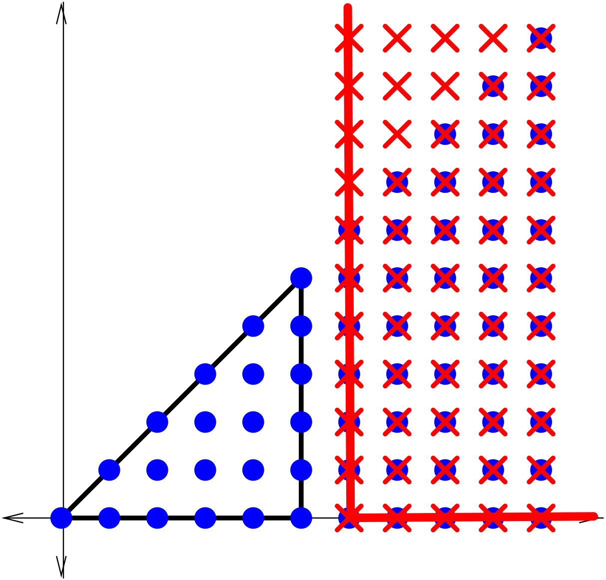


$$(1+x+x^2+x^3+\cdots)$$

$$\cdot(1+xy+(xy)^2+\cdots)$$

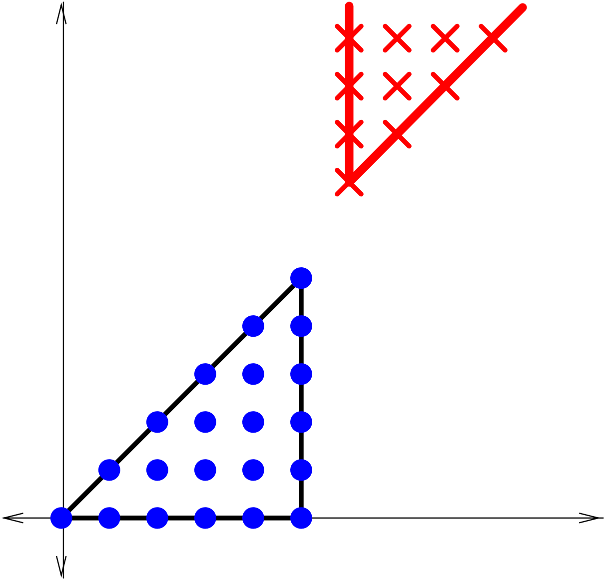
$$= \frac{1}{(1-x)(1-xy)}$$

A Computer Example



$$-x^6 \cdot (1 + x + x^2 + \cdots) \cdot (1 + y + y^2 + \cdots) = -\frac{x^6}{(1-x)(1-y)}$$

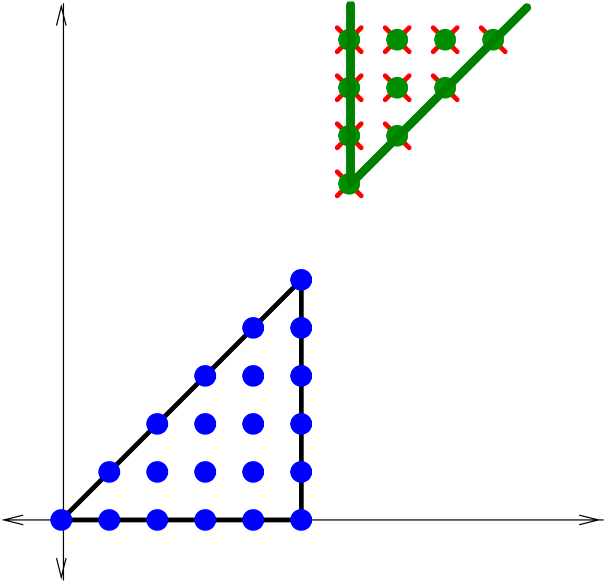
A Computer Example



$$\frac{1}{(1-x)(1-xy)}$$

$$-\frac{x^6}{(1-x)(1-y)}$$

A Computer Example



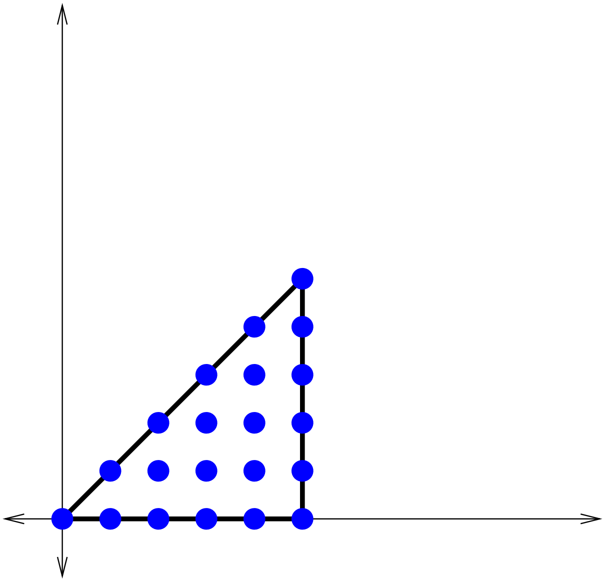
$$+x^6y^7$$

$$\cdot(1+xy+(xy)^2+\cdots)$$

$$\cdot(1+y+y^2+\cdots)$$

$$=\frac{x^6y^7}{(1-xy)(1-y)}$$

A Computer Example



$$f(S;x,y) = \frac{1}{(1-x)(1-xy)} - \frac{x^6}{(1-x)(1-y)} + \frac{x^6y^7}{(1-xy)(1-y)}.$$

A Computer Example

We have

$$f(S;x,y) = \sum_{(i,j)\in S} x^i y^j.$$

We want

$$\sum_{(i,j)\in S} ij.$$

A Computer Example

We have

$$f(S;x,y) = \sum_{(i,j)\in S} x^i y^j.$$

We want

$$\sum_{(i,j)\in S} ij.$$

$$\frac{\partial^2}{\partial x \partial y} f(S;x,y) = \sum_{(i,j)\in S} ij x^{i-1} y^{j-1}.$$

Therefore we want

$$\frac{\partial^2}{\partial x \partial y} f(S;x,y)\Big|_{x=1,y=1} = 140.$$

Summary

- We can often use patterns in seemingly complicated sets to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sets.

Quick now!

Question: When can we find $f(S; \mathbf{x})$ quickly?

We want an algorithm that inputs a Presburger sentence and outputs $f(S; \mathbf{x})$.

The input size is the number of bits needed to encode the input for the algorithm.

The input size of a number a is approximately

$$\log_2(a).$$

An algorithm is polynomial time if there is a polynomial p such that the algorithm runs in at most $p(\text{input size})$ steps.

polynomial time = quick

Quick now!

Question: When can we find $f(S; \mathbf{x})$ quickly?

We want an algorithm that inputs a Presburger sentence and outputs $f(S; \mathbf{x})$.

The input size is the number of bits needed to encode the input for the algorithm.

The input size of a number a is approximately

$$\log_2(a).$$

An algorithm is polynomial time if there is a polynomial p such that the algorithm runs in at most $p(\text{input size})$ steps.

polynomial time = quick

Quick now!

Question: When can we find $f(S; \mathbf{x})$ quickly?

We want an algorithm that inputs a Presburger sentence and outputs $f(S; \mathbf{x})$.

The input size is the number of bits needed to encode the input for the algorithm.

The input size of a number a is approximately

$$\log_2(a).$$

An algorithm is polynomial time if there is a polynomial p such that the algorithm runs in at most p(input size) steps.

polynomial time = quick

Good Algorithms

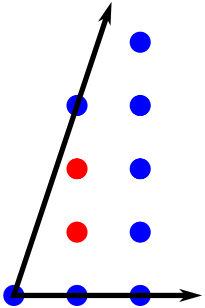
- If there are no quantifiers, there is a polynomial time algorithm (if we fix the number of variables) [Barvinok].
- If only $\exists$'s are needed to define S, there is a polynomial time algorithm (if we fix the number of variables and linear inequalities) [W].

No Quantifiers

This is like the previous example:

$$S = \{(i, j) \in \mathbb{N}^2 \mid i \leq 5 \text{ and } j \leq i\}.$$

- Inclusion-Exclusion of cones [Brion]
- Not all cones are "nice" (unimodular):

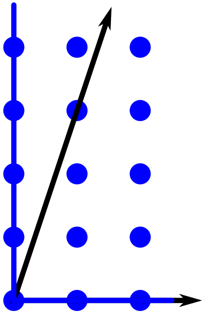


No Quantifiers

This is like the previous example:

$$S = \{(i, j) \in \mathbb{N}^2 \mid i \leq 5 \text{ and } j \leq i\}.$$

- Inclusion-Exclusion of cones [Brion]
- Not all cones are "nice" (unimodular):



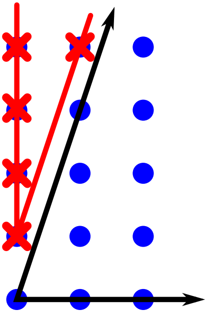
unimodular decomposition [Barvinok]

No Quantifiers

This is like the previous example:

$$S = \{(i,j) \in \mathbb{N}^2 \mid i \leq 5 \text{ and } j \leq i\}.$$

- Inclusion-Exclusion of cones [Brion]
- Not all cones are "nice" (unimodular):



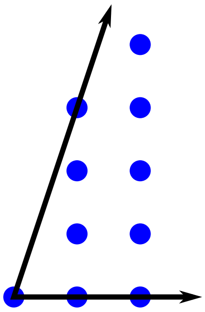
unimodular decomposition [Barvinok]

No Quantifiers

This is like the previous example:

$$S = \{(i, j) \in \mathbb{N}^2 \mid i \leq 5 \text{ and } j \leq i\}.$$

- Inclusion-Exclusion of cones [Brion]
- Not all cones are "nice" (unimodular):



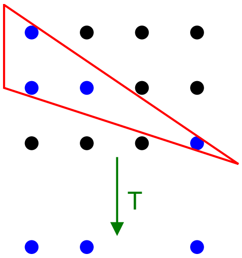
unimodular decomposition [Barvinok]

Existential Quantifiers

Projections

$$S = \{ i \in \mathbb{N} \;\Big|\; \exists j \in \mathbb{N} : \; (i, j) \in P \}.$$

We need to compute generating functions for projections of $P \cap \mathbb{Z}^n$, where P is a polyhedron.



$T(i, j) = i$, and $S = T(P \cap \mathbb{Z}^2)$.

Existential Quantifiers

Example: Frobenius Problem with $a_1 = 2$, $a_2 = 5$.

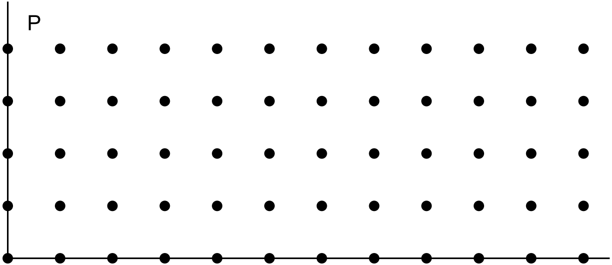
$P = \{(i, j) : \; i, j \geq 0\}$

$T(i, j) = 2i + 5j$. (1-d Kernel)

Then $S = T(P \cap \mathbb{Z}^2)$.

Existential Quantifiers

1-d Kernel



Compute $f(P \cap \mathbb{Z}^2; x, y) = \frac{1}{(1-x)(1-y)}$. [Barvinok]

Compute $f(P \cap \mathbb{Z}^2; t^2, t^5)$. Then $x^i y^j \mapsto t^{2i+5j}$.

Existential Quantifiers

1-d Kernel

$$f(P \cap \mathbb{Z}^2; t^2, t^5) = \frac{1}{(1-t^2)(1-t^5)} = (1 + t^2 + t^4 + \cdots)(1 + t^5 + \cdots)$$
$$= 1 + t^2 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + \cdots$$

.

Existential Quantifiers

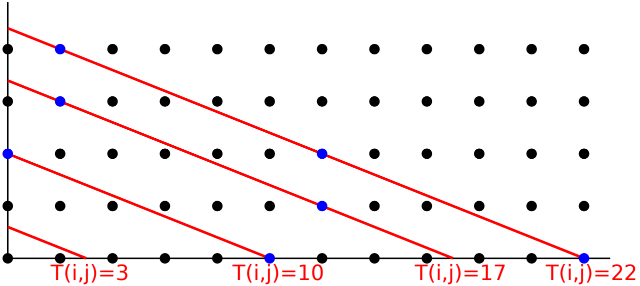
1-d Kernel

$$f(P \cap \mathbb{Z}^2; t^2, t^5) = \frac{1}{(1-t^2)(1-t^5)} = (1 + t^2 + t^4 + \cdots)(1 + t^5 + \cdots)$$
$$= 1 + t^2 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + 2t^{10} + \cdots$$

.

Existential Quantifiers

1-d Kernel

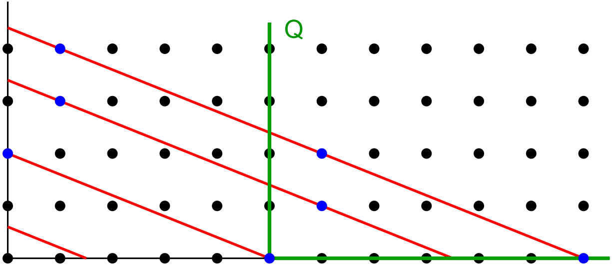


$$f(P \cap \mathbb{Z}^2; t^2, t^5) = \frac{1}{(1-t^2)(1-t^5)} = (1 + t^2 + t^4 + \cdots)(1 + t^5 + \cdots)$$
$$= 1 + t^2 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + 2t^{10} + \cdots$$

Problem: T is not 1-1 on $P \cap \mathbb{Z}^2$.

Existential Quantifiers

1-d Kernel



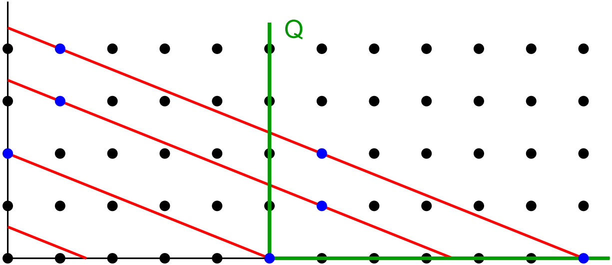
Let $Q = \{(i,j) : \; i \geq 5, j \geq 0\}$.

$$f(Q \cap \mathbb{Z}^2; x, y) = \frac{x^5}{(1-x)(1-y)}.$$

$$f(Q \cap \mathbb{Z}^2; t^2, t^5) = \frac{t^{10}}{(1-t^2)(1-t^5)}.$$

Existential Quantifiers

1-d Kernel

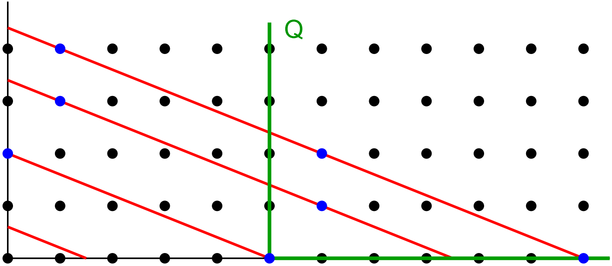


T is 1-1 on $(P - Q) \cap \mathbb{Z}^2$.

$$f(S;t) = f(P \cap \mathbb{Z}^2; t^2, t^5) - f(Q \cap \mathbb{Z}^2; t^2, t^5)$$
$$= \frac{1 - t^{10}}{(1 - t^2)(1 - t^5)}.$$

Existential Quantifiers

1-d Kernel

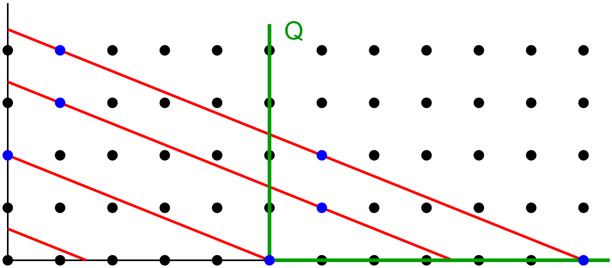


T is 1-1 on $(P - Q) \cap \mathbb{Z}^2$.

$$f(S;t) = f(P \cap \mathbb{Z}^2; t^2, t^5) - f(Q \cap \mathbb{Z}^2; t^2, t^5)$$
$$= \frac{1 - t^{10}}{(1 - t^2)(1 - t^5)}.$$

Existential Quantifiers

1-d Kernel



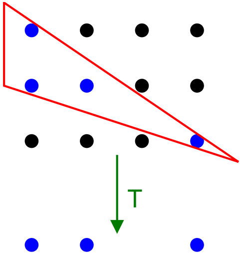
Why This Works: There are no gaps in the fibers of T.

Only works for 1-d kernel.

Existential Quantifiers

Higher-d Kernel

General situation: Use induction on the dimension of the kernel.

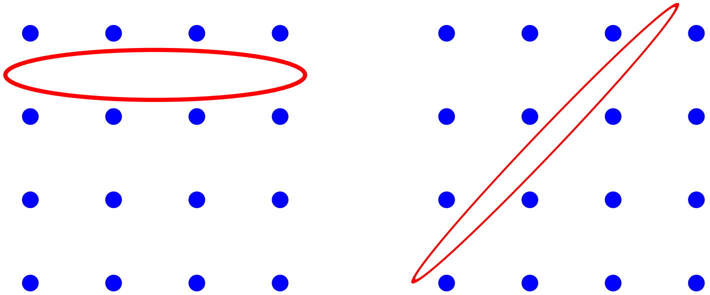


Must control the gaps.

Existential Quantifiers

Higher-d Kernel

A Tool:

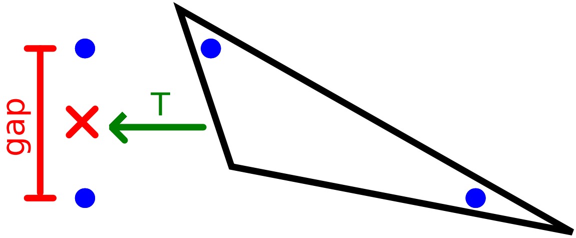


Flatness Theorem (Khinchin): Convex objects that contain no integer points are thin in some direction.

Existential Quantifiers

Higher-d Kernel

Looking at a fiber of the desired projection, suppose we project onto the thinnest direction.

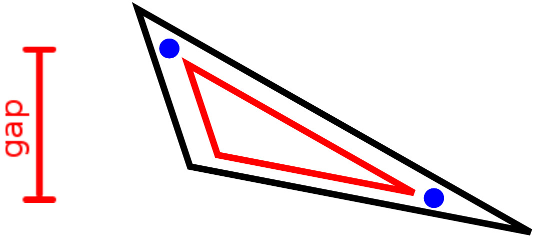


If there are large gaps,

Existential Quantifiers

Higher-d Kernel

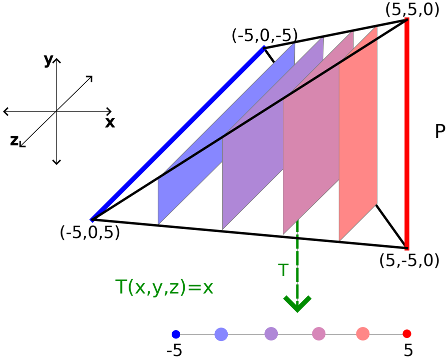
Looking at a fiber of the desired projection, suppose we project onto the thinnest direction.



If there are large gaps,
Then there is a lattice-free polytope that is wide.
Contradiction.

Existential Quantifiers

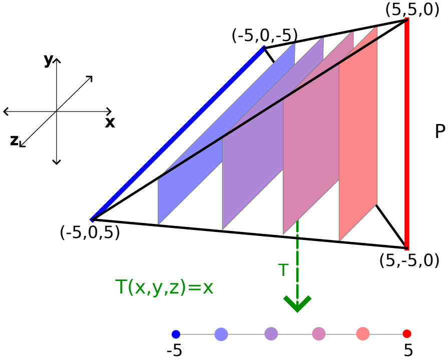
Higher-d Kernel



Look at a fiber of $T(P)$, and pick the thinnest direction. That direction gets projected out last (inductively).

Existential Quantifiers

Higher-d Kernel



Complication: Different fibers have different thin directions.
Solution: Break things up into pieces [Kannan].

Applications

- Frobenius problem [Barvinok-W]
- Minimal Hilbert Bases [Barvinok-W]
- Hilbert series of rings generated by monomials [Barvinok-W]
- Test sets for integer programming [Barvinok-W]
- Integer programming gaps [Hoşten-Sturmfels]
- Reduced Gröbner bases for toric ideals, and some related computations [De Loera, et al.]
- Standard pairs and arithmetic degree of order ideals in integer programming [Thomas-W]
- Ehrhart quasi-polynomials (and their period) [W]

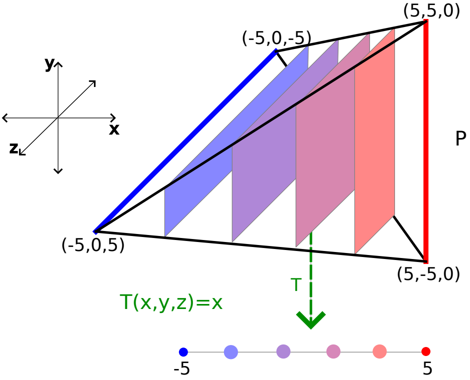
Summary

- We can often use hidden structure in seemingly complicated sets to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sets.

Summary

- We can often use hidden structure in seemingly complicated sets to encode them compactly as generating functions.
- We can manipulate the generating functions to answer questions about the sets.
- We can do many of these things quickly.

Thank You!



The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.

  Bad

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.

  Bad

- Fix number of variables, no quantifiers.

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.
  Bad
- Fix number of variables, no quantifiers.
  Good [Barvinok]

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.
  Bad
- Fix number of variables, no quantifiers.
  Good [Barvinok]
- Fix number of variables, quantifiers allowed.

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.
  Bad
- Fix number of variables, no quantifiers.
  Good [Barvinok]
- Fix number of variables, quantifiers allowed.
  Bad, even with a single quantifier [W; Schöning]

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.
  Bad
- Fix number of variables, no quantifiers.
  Good [Barvinok]
- Fix number of variables, quantifiers allowed.
  Bad, even with a single quantifier [W; Schöning]
- Fix number of variables and inequalities, only $\exists$ quantifiers.

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.
  Bad
- Fix number of variables, no quantifiers.
  Good [Barvinok]
- Fix number of variables, quantifiers allowed.
  Bad, even with a single quantifier [W; Schöning]
- Fix number of variables and inequalities, only $\exists$ quantifiers.
  Good [W]

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.
  Bad
- Fix number of variables, no quantifiers.
  Good [Barvinok]
- Fix number of variables, quantifiers allowed.
  Bad, even with a single quantifier [W; Schöning]
- Fix number of variables and inequalities, only $\exists$ quantifiers.
  Good [W]
- Fix number of variables and inequalities, mixed quantifiers.

The Good, the Bad, and the ____

Presburger sentences from an algorithmic perspective:

- General sentences.
  Bad
- Fix number of variables, no quantifiers.
  Good [Barvinok]
- Fix number of variables, quantifiers allowed.
  Bad, even with a single quantifier [W; Schöning]
- Fix number of variables and inequalities, only $\exists$ quantifiers.
  Good [W]
- Fix number of variables and inequalities, mixed quantifiers.
  ?????

Thank You!