Squooshing the Cube

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*Hugely indebted to Igor Pak's "Inflating polyhedral surfaces", preprint (2006), for ideas and pictures

Squooshing

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Okay, not really. But squoosh is in the dictionary. It rhymes with push.

We will take it to mean, "You can bend, but you can't stretch."

Challenge: Squoosh a cube and increase its volume.

Why it can't be done:

Ancient History

Euclid of Alexandria (330-260 BC)

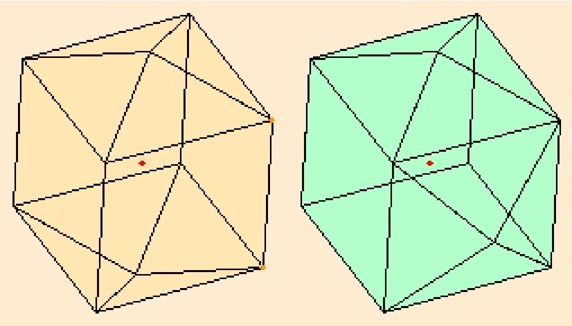
"Things which coincide are equal."

"Equal and similar solid figures are those contained by similar planes equal in multitude and magnitude."

Seems to mean: two polyhedra are the same if the faces of one are congruent to the faces of the other.

Ancient History

Are these equal?[1]



[1] None of the pictures are mine. I'm lazy.

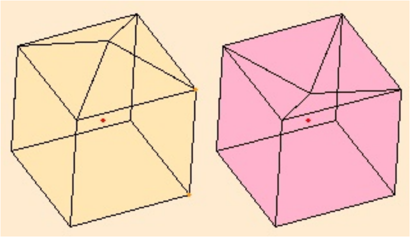
Ancient History

Hero(n) of Alexandria (10-70 AD)

Added that the faces must be similarly situated to one another.

Ancient History

Are these equal?



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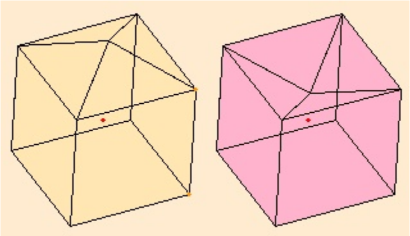
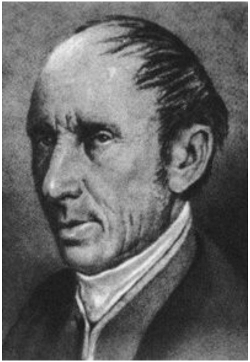


Only count convex polyhedra.

Is there a convex analog of Steffan's flexible polyhedron?

Rigor!

Augustin Cauchy (1789-1857).



"Modern mathematics is indebted to Cauchy for two of its major interests, each of which marks a sharp break with the mathematics of the eighteenth century. The first was the introduction of rigor into mathematical analysis. The second thing of fundamental importance was on the opposite side – the combinatorial."
(E.T. Bell)

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Alexandrov's Theorem (1940's): Squoosh a convex polyhedron, and it will no longer be convex.

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Why it can't be done:

- A sphere is the shape that encloses the largest volume for a given surface area.
- A sphere is Perfection. Very convex.
- Squooshing a cube makes it non-convex.
- That surely can't increase the volume.

Why it can be done:

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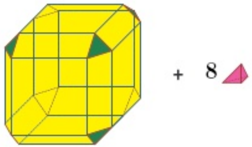
Do it!

Igor Pak (based on Milka): Flatten the edges.

Volume Calculations

1x1x1 cube. Let $2x$ be the width of the flattened edges.
Let $d = 2x/\sqrt{2} = x\sqrt{2}$.

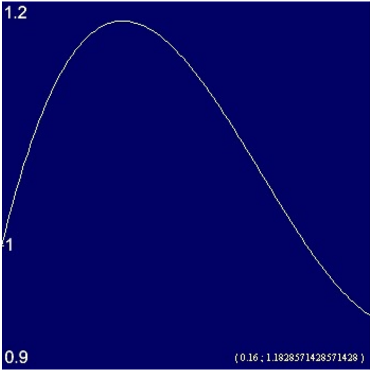


Shape	Volume (each)
one inner cube	$(1-2x)^3$
six slabs	$(1-2x)^2 d$
twelve triangular prisms	$\frac{d^2}{2}(1-2x)$
sixteen pyramids	$\frac{d^3}{6}$

Total Volume: $(1-2x)^3 + 6(1-2x)^2 d + 12\frac{d^2}{2}(1-2x) + 16\frac{d^3}{6}$.

Volume Calculations

For $0 \leq x \leq 0.5$,



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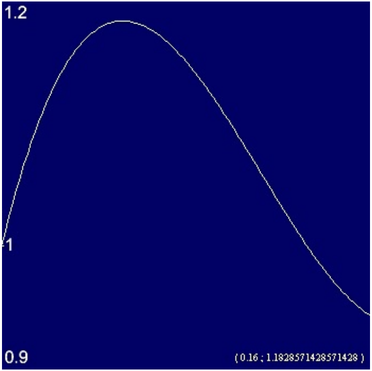

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When $x = 0.5$, get the stellated octahedron, volume 0.95.

Beyond

Open question: What is the maximum volume we can squoosh the cube to?

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Know max is not achieved by something with flat sides.

Anything with flat sides can be squooshed bigger (Pak).

Balloons!

Mylar balloon: 2 flat circles glued on edges – Volume zero.

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Wildly non-convex. Crimping.

Balloons!

If volume of the mylar balloon maximized:

Look at arbitrarily small piece of surface area:
must be crimps on that piece.

Near glued edges, $\approx 10\%$ of surface area lost in crimps.
(Paulson)

Pillows!

A square pillow cushion:

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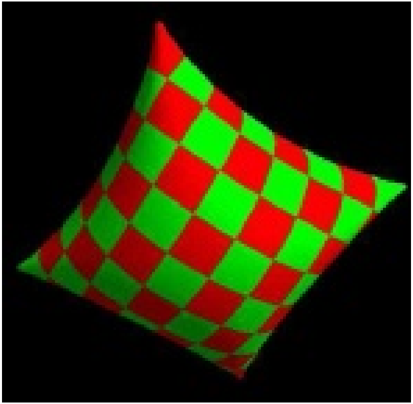

Corners stick out (non-convex) and crimping.

Cubes!

And finally, our Cube.

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