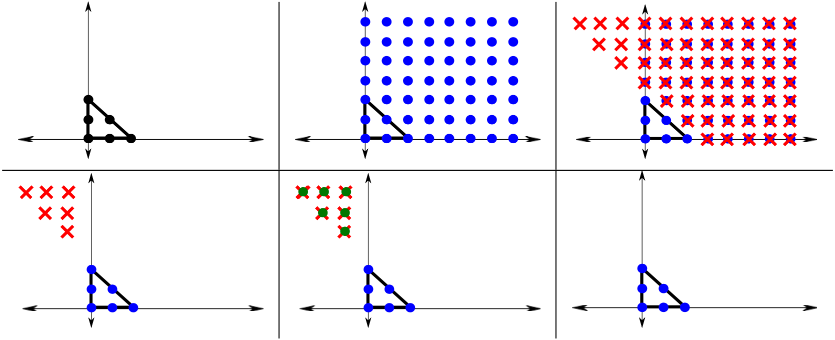
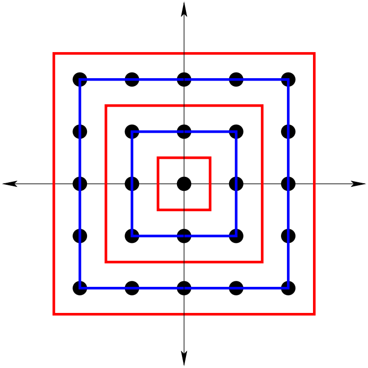
Counting with Rational Generating Functions

Kevin Woods, Oberlin College
(joint work with Sven Verdoolaege, Universiteit Leiden)



Counting Problems

Example: $P = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subseteq \mathbb{R}^2$.



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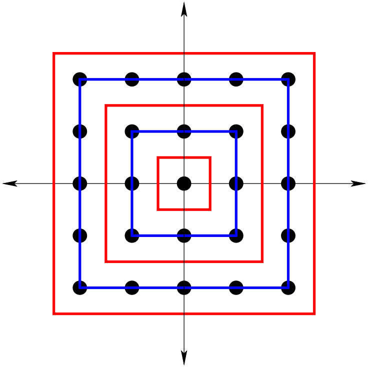

Define $c(t) = \#\{tP \cap \mathbb{Z}^2\}$, for $t \in \mathbb{Z}_+$.

$$c(t) = \begin{cases} (t+1)^2, & \text{for } t \text{ even} \\ t^2, & \text{for } t \text{ odd} \end{cases} = \left(2\left\lfloor \frac{t}{2} \right\rfloor + 1\right)^2.$$

Counting Problems

Define $c(s,t)$ by

$$\sum_{s,t} c(s,t) z^s w^t = \frac{1}{(1-zw)(1-z^2w)(1-z)(1-w)}..$$

Is there a "nice" formula for $c(s,t)$?
This talk will focus on finding one.

An Example

$$\sum_t c(t) z^t = \frac{1}{(1-z)^3} = (1 + z + \cdots)(1 + z + \cdots)(1 + z + \cdots)$$

Let's compute $c(t)$.

$$\begin{aligned} c(t) &= \#\{a, b, c \in \mathbb{Z} : \; a, b, c \geq 0, \; a + b + c = t\} \\ &= \#\{a, b \in \mathbb{Z} : \; a, b \geq 0, \; a + b \leq t\} \\ &= \#(P_t \cap \mathbb{Z}^2), \end{aligned}$$

for some polytope P_t.

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An Example

Idea:

- For fixed t, look at

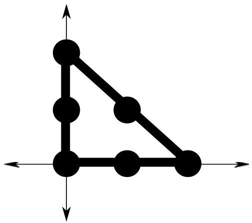
$$\sum_{(a,b)\in P_t\cap\mathbb{Z}^2} x^{a} y^{b}.$$

- Plug in $x = y = 1$.
- Investigate what happens as t changes.

An Example

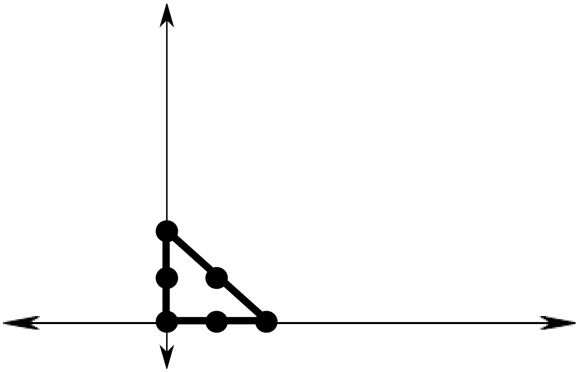
$$c(t) = \#\{a, b \in \mathbb{Z} : \ a, b \geq 0, \ a + b \leq t\}.$$

Example: $t = 2$.



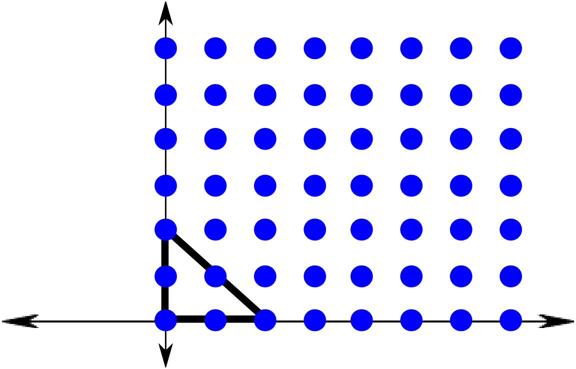
$$c(2) = x^0y^0 + x^1y^0 + x^2y^0 + x^0y^1 + x^1y^1 + x^0y^2\Big|_{x=y=1} = 6.$$

An Example



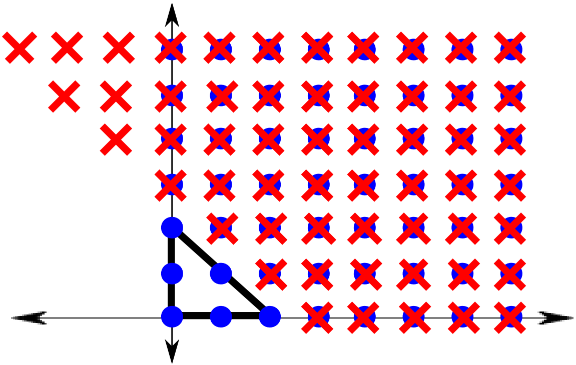
What happens when t changes?

An Example



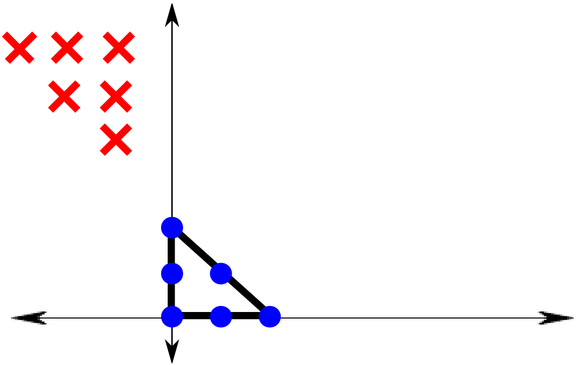
$$(1 + x + x^2 + \cdots)$$
$$\cdot(1+y+y^2+\cdots)$$
$$= \frac{1}{(1-x)(1-y)}$$

An Example



$$-\frac{x^{t+1}}{(1-x)(1-x^{-1}y)}$$

An Example



$$\frac{1}{(1-x)(1-y)}$$

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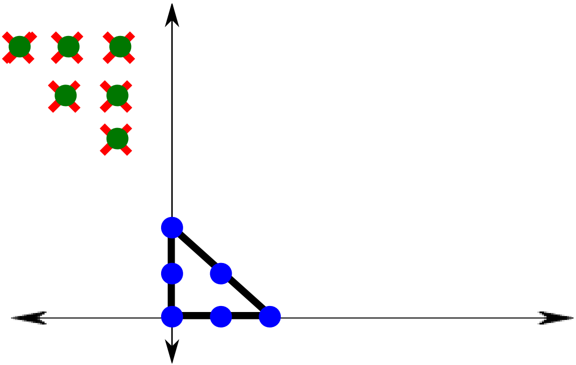

$$+\frac{x^{-1}y^{t+2}}{(1-x^{-1}y)(1-y)}$$

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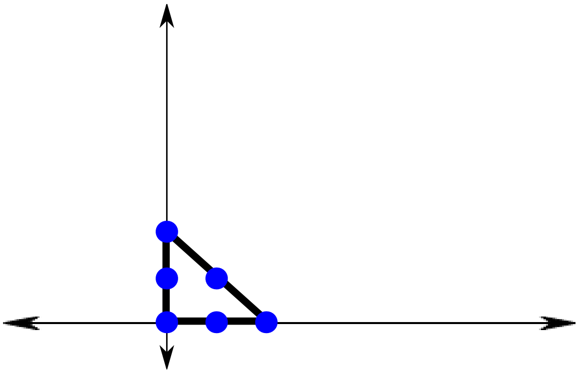

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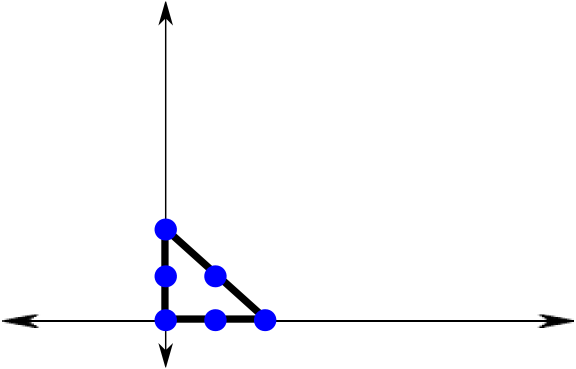

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$$-\frac{x^{t+1}}{(1-x)(1-x^{-1}y)}$$

$$+\frac{x^{-1}y^{t+2}}{(1-x^{-1}y)(1-y)}$$

This is Brion's Theorem.

Note: t only appears in exponents of numerators!

An Example

$$\frac{1}{(1-x)(1-y)} - \frac{x^{t+1}}{(1-x)(1-x^{-1}y)} + \frac{x^{-1}y^{t+2}}{(1-x^{-1}y)(1-y)}.$$

We've found the generating function. Now plug in $x = 1$, then $y = 1$.

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But $x = 1$ is a pole of the first term!

When summed, poles must cancel.

An Example

For each term f, need to find a_0.

$$\begin{aligned}
f &= a_{-1}(x-1)^{-1} + a_0 + a_1(x-1) + \cdots \\
(x-1)f &= a_{-1} + a_0(x-1) + a_1(x-1)^2 + \cdots \\
\frac{\partial}{\partial x}(x-1)f &= a_0 + 2a_1(x-1) + \cdots \\
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An Example

2nd term:

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\begin{aligned}
f &= -\frac{x^{t+1}}{(1-x)(1-x^{-1}y)} \\
(x-1)f &= \frac{x^{t+1}}{1-x^{-1}y} \\
\frac{\partial}{\partial x}(x-1)f &= \frac{(t+1)x^{t}\cdot(1-x^{-1}y) - x^{-2}y\cdot x^{t+1}}{(1-x^{-1}y)^2} \\
\frac{\partial}{\partial x}(x-1)f\Big|_{x=1} &= \frac{(t+1)(1-y)-y}{(1-y)^2}.
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Taking the derivative creates polynomials in t.

An Example

Putting the three terms together, we have

$$\frac{(t+1)(1-y)+y-y^{t+2}}{(1-y)^2}.$$

Plugging in $y=1$ as well, the final answer is

$$\frac{(t+1)(t+2)}{2}.$$

An Example

Recap:

- Find the generating function. Exponentials in numerator are linear functions of t. Everything else is constant with t.
- Plug in $x = y = 1$. Taking derivatives creates a polynomial in t.

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- Plug in $x = y = 1$. Taking derivatives creates a polynomial in t.

This works in general.

Liar!

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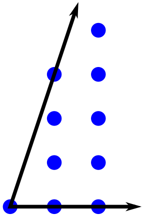
This example is misleadingly simple.

Lie 1

Not all cones have such nice generating functions, only unimodular cones do.



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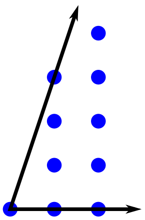

Barvinok: Any cone can be written as a sum and difference of unimodular cones (in polynomial time).

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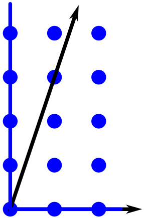

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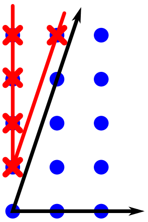

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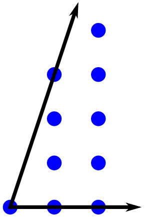

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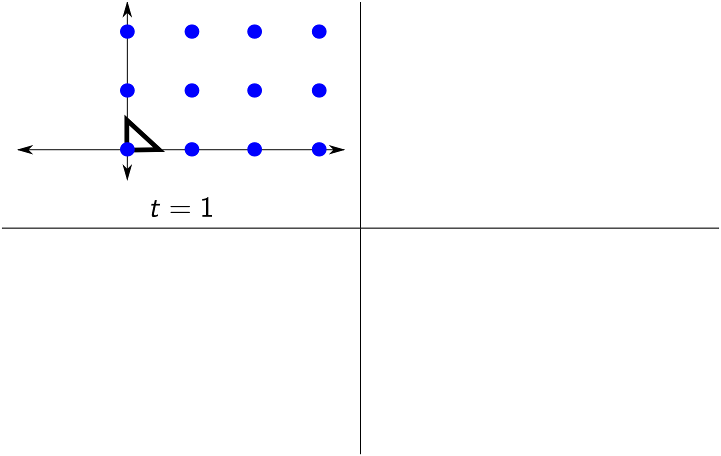
Lie 2

The vertices of the polytopes are not always integral.

$$\frac{1}{(1-z)(1-z^2)^2}.$$



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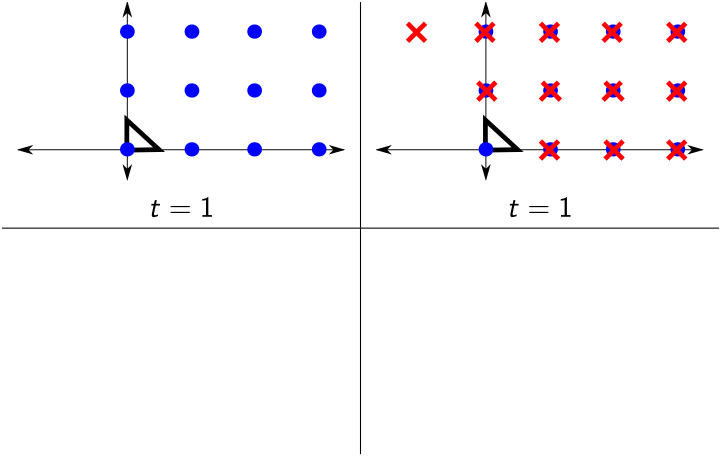

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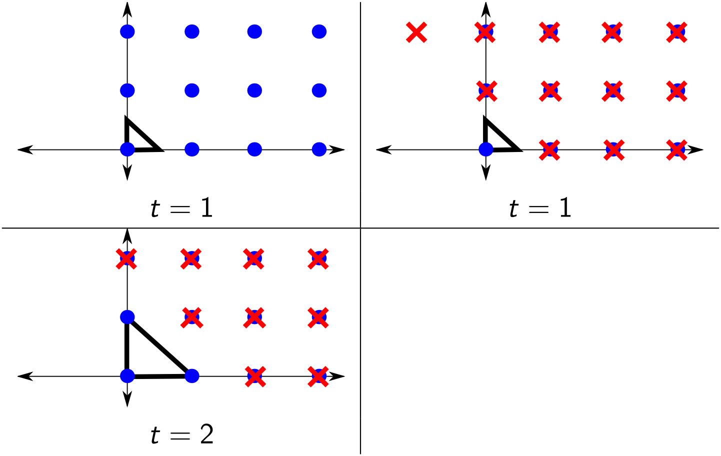

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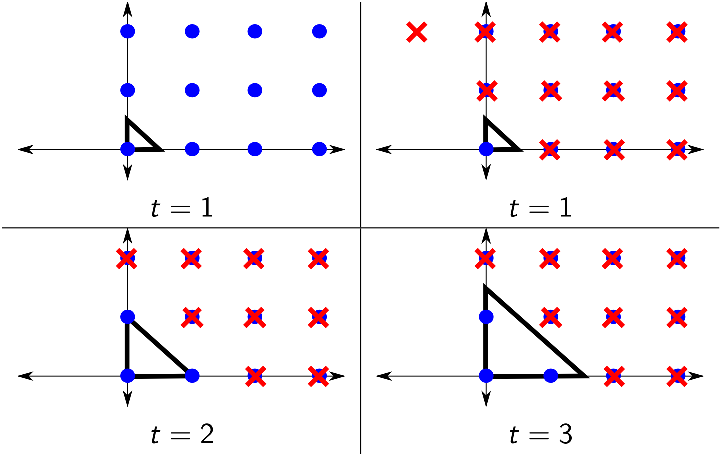


vertex of red cone is $\left(\left\lfloor \frac{t+2}{2} \right\rfloor, 0\right)$.

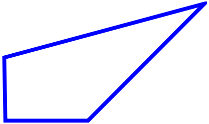
Lie 3

With more than one parameter, vertices may disappear.

$$\sum_{s,t} c(s,t) z^s w^t = \frac{1}{(1-zw)(1-z^2w)(1-z)(1-w)}.$$

Corresponding polytope is

$$\{a, b \in \mathbb{Z} : \ a \geq 0, \ b \geq 0, \ 2b - a \leq 2t - s, \ a - b \leq s - t\}.$$



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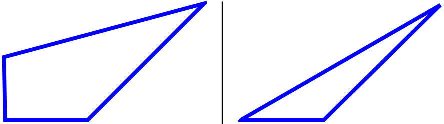

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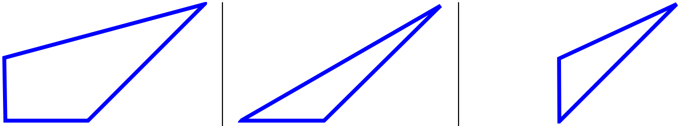

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Lie 3

End up with

$$c(s,t) = \begin{cases} \frac{s^2}{2} - \lfloor \frac{s}{2} \rfloor s + \frac{s}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1 & \text{if } t \le s \le 2t \\ st - \lfloor \frac{s}{2} \rfloor s - \frac{t^2}{2} + \frac{t}{2} + \lfloor \frac{s}{2} \rfloor^2 + \lfloor \frac{s}{2} \rfloor + 1 & \text{if } 0 \le 2t \le s \\ \frac{t^2}{2} + \frac{3t}{2} + 1 & \text{if } 0 \le s \le t \end{cases}.$$

Example courtesy of Sven Verdoolaege's `barvinok`.

Two Sides of the Same Coin

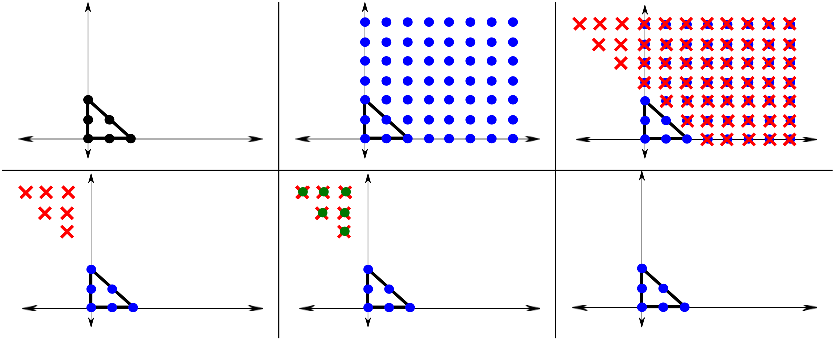
Heads: The nimble a Rational Generating Function

$$\sum_{s,t} c(s,t) z^s w^t$$

Tails: The concrete "piecewise step-polynomial".

You don't have to choose your favorite representation.
You can translate back and forth in polynomial time.
[Verdoolaege, W]

Thank You!



Two Sides of the Same Coin

Heads: a Rational Generating Function.

A function in the form

$$f(\mathbf{z}) = \sum_{i \in I} \alpha_i \frac{\mathbf{z}^{\mathbf{p_i}}}{(1-\mathbf{z}^{\mathbf{b_{i1}}})(1-\mathbf{z}^{\mathbf{b_{i2}}})\cdots(1-\mathbf{z}^{\mathbf{b_{ik_i}}})},$$

where $\mathbf{z} \in \mathbb{C}^n$, I is a finite set, $\alpha_i \in \mathbb{Q}$, $\mathbf{p_i} \in \mathbb{Z}^\mathbf{n}$, and $\mathbf{b_{ij}} \in \mathbb{Z}^\mathbf{n} \setminus \{\mathbf{0}\}$.

Two Sides of the Same Coin

Tails: a Piecewise Step-polynomial.

Defined piecewise on polyhedral regions Q_i:

$c(\mathbf{s}) = c_i(\mathbf{s})$, for $\mathbf{s} \in Q_i$, where

$$c_i(\mathbf{s}) = \sum_{j=1}^{m} \alpha_{ij} \prod_{k=1}^{d_{ij}} \lfloor p_{ijk}(\mathbf{s}) \rfloor$$

with $\alpha_{ij} \in \mathbb{Q}$ and p_{ijk} are degree one polynomials over $\mathbb{Q}$.

Two Sides of the Same Coin

- Rational generating functions are nimble.
- Piecewise Step-polynomials are concrete.

Two Sides of the Same Coin

Theorem

Fix d and k. There is a polynomial time algorithm that:

- *Given a rational generating function $f(\mathbf{x})$ in d variables, with at most k binomials in each denominator, computes the piecewise step polynomial $c(\mathbf{s})$ such that*

$$f(\mathbf{x}) = \sum_{\mathbf{s}} c(\mathbf{s})\mathbf{x}^{\mathbf{s}}.$$

- *Given a piecewise step-polynomial $c : \mathbb{Z}^d \to \mathbb{Z}$ of degree at most k, computes the rational generating function f.*

Two Sides of the Same Coin

Given a rational generating function,

$$f(\mathbf{z}) = \sum_{i \in I} \alpha_i \frac{\mathbf{z}^{\mathbf{p_i}}}{(1-\mathbf{z}^{\mathbf{b_{i1}}})(1-\mathbf{z}^{\mathbf{b_{i2}}})\cdots(1-\mathbf{z}^{\mathbf{b_{ik_i}}})},$$

- Each term is a vector partition function. Compute each, and combine at the end.
- Divide parameter space into pieces, based on the vertices of the corresponding polytope, and compute it for each piece.
- Brion: Break up into cones. The vertex of each cone is the floor of a linear function.
- Barvinok: Decompose each cone into unimodular cones.
- Plug in $\mathbf{x} = \mathbf{1}$. Taking derivatives to do this will create step-polynomials.

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Given a rational generating function,

$$f(\mathbf{z}) = \sum_{i \in I} \alpha_i \frac{\mathbf{z}^{\mathbf{p_i}}}{(1-\mathbf{z}^{\mathbf{b_{i1}}})(1-\mathbf{z}^{\mathbf{b_{i2}}})\cdots(1-\mathbf{z}^{\mathbf{b_{ik_i}}})},$$

- Each term is a vector partition function. Compute each, and combine at the end.
- Divide parameter space into pieces, based on the vertices of the corresponding polytope, and compute it for each piece.
- Brion: Break up into cones. The vertex of each cone is the floor of a linear function.
- Barvinok: Decompose each cone into unimodular cones.
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Note: This proof

- Shows that $c(s)$ is a piecewise quasi-polynomial.
- Gives an explicit formula for $c(s)$.
- Computes a concise formula quickly (in polynomial time).

Two Sides of the Same Coin

Conversely, given a piecewise step-polynomial,

- For each step-monomial, create a polytope $P \subseteq \mathbb{R}^d \times \mathbb{R}^k$ such that
$$c(\mathbf{s}) = \#\{\mathbf{a} \in \mathbb{Z}^k : (\mathbf{s}, \mathbf{a}) \in P\}.$$
- Find the rational generating function
$$f(\mathbf{z}, \mathbf{x}) = \sum_{(\mathbf{s},\mathbf{a}) \in P \cap \mathbb{Z}^{d+k}} \mathbf{z}^{\mathbf{s}} \mathbf{x}^{\mathbf{a}}.$$
- Compute
$$f(\mathbf{z}, \mathbf{1}) = \sum_{\mathbf{s}} c(\mathbf{s}) \mathbf{z}^{\mathbf{s}}.$$

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Thank You!