Exercise 1 solution: Let $r_\omega : \mathbf{T} \to \mathbf{T}$, $r_\omega(\theta) = \theta + \omega$. Note that $r_\omega^n(\theta) = \theta + n\omega$.

(a) Let p/q be rational. Then $r_{p/q}^q(\theta) = \theta + q \cdot p/q = \theta + p$. Since p is an integer, θ and $\theta + p$ are the same point on $\mathbf{T}$, and $r_{p/q}^q(\theta) = \theta$. Thus all points are periodic under $r_{p/q}^q$ with period q.

(b) Suppose ω is irrational. Let $\theta \in \mathbf{T}$. Suppose $r_\omega^n(\theta) = r_\omega^m(\theta)$. Then $\theta + n\omega = \theta + m\omega + k$ for some integer k, implying $(n-m)\omega = k$. Since ω is irrational, $n = m$. Hence $o(\theta, r_\omega)$ is an infinite subset of $\mathbf{T}$.

Let $\epsilon > 0$ be given. Since $\mathbf{T}$ is bounded, the Bolzano-Weierstrass Theorem implies $o(\theta, r_\omega)$ has a limit point in $\mathbf{T}$. Hence there exist integers $n > m$ with $|r_\omega^n(\theta) - r_\omega^m(\theta)| < \epsilon$. Letting $k = n - m$, we have $|r_\omega^k(\theta) - \theta| < \epsilon$ since r_ω preserves lengths. Now $r_\omega^k(\theta)$ maps the arc connecting θ and $r_\omega^k(\theta)$ to the arc connecting $r_\omega^k(\theta)$ and $r_\omega^{2k}(\theta)$, and this arc has length less than ϵ. The set $\{\theta, r_\omega^k(\theta), r_\omega^{2k}(\theta), \ldots\}$ thus partitions $\mathbf{T}$ into arcs of length less than ϵ, so that $o(\theta, r_\omega)$ is dense in $\mathbf{T}$. (This argument is from *An Introduction To Chaotic Dynamical Systems* (2nd edition) by R.L. Devaney, p. 22.)