Exercise 11 solution:

(a) Suppose there exist integers p, q and a real x satisfying $F^q(x) = x + p$. then

$$\lim_{m\to\infty} \frac{F^{mq}(x) - x}{mq} = \lim_{m\to\infty} \frac{x + mp - x}{mq} = \frac{p}{q}.$$

Let $0 \leq r < q$ be an integer, and let $y = k + \langle y \rangle$, k an integer. Then $F(y) - y = F(k + \langle y \rangle) - (k + \langle y \rangle) = F(\langle y \rangle) - \langle y \rangle$, so when considering $F(y) - y$ it suffices to take $y \in [0, 1]$. Since F^r is continuous, it is bounded on $[0, 1]$. Hence there exists a constant M such that $|F^r(y) - y| \leq M$ for all $y \in \mathbf{R}$ and all $0 \leq r < q$.

Let $n \in \mathbf{Z}$, and write $n = mq + r$, $0 \leq r < q$. Pick the constant M as above. Then

$$\frac{F^n(x) - x}{n} = \frac{F^{mq+r}(x) - x}{mq + r} = \frac{F^{mq+r}(x) - F^{mq}(x)}{mq + r} + \frac{F^{mq}(x) - x}{mq + r}$$

$$= \frac{mq}{mq + r} \cdot \frac{F^{mq}(x) - x}{mq} + \frac{F^r(F^{mq}(x)) - F^{mq}(x)}{mq + r}.$$

Note that

$$\left| \frac{F^r(F^{mq}(x)) - F^{mq}(x)}{mq + r} \right| \leq \frac{M}{n},$$

and hence approaches 0 as $n \to \infty$. Since $0 \leq r < q$, $mq/(mq + r) \to 1$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n} = \lim_{m\to\infty} \frac{F^{mq}(x) - x}{mq} = \frac{p}{q}.$$

We have shown that if F has a p/q-periodic point then the rotation number is the rational p/q.

(b) Suppose F has no p/q-periodic points. Fix $n > 0$. Then for all $x \in \mathbf{R}$, and for all integers k, $F^n(x) - x \neq k$. So by the Intermediate Value Theorem, there is an integer k_n such that $k_n < F^n(x) - x < k_n + 1$ for all $x \in \mathbf{R}$. Applying this inequality with x-values of $0, F^n(0), F^{2n}(0), \ldots$, we have

$$k_n < F^n(0) < k_n + 1$$

$$k_n < F^{2n}(0) - F^n(0) < k_n + 1$$

$$\vdots$$

$$k_n < F^{mn}(0) - F^{(m-1)n}(0) < k_n + 1.$$

Adding these inequalities yields $mk_n < F^{mn}(0) < m(k_n + 1)$, or

$$\frac{k_n}{n} < \frac{F^{mn}(0)}{mn} < \frac{k_n}{n} + \frac{1}{n}. \quad \text{Also, } \frac{k_n}{n} < \frac{F^n(0)}{n} < \frac{k_n}{n} + \frac{1}{n},$$

so that

$$\left| \frac{F^{mn}(0)}{mn} - \frac{F^n(0)}{n} \right| < \frac{1}{n}.$$

Interchanging the roles of n and m, we also have

$$\left| \frac{F^{mn}(0)}{mn} - \frac{F^m(0)}{m} \right| < \frac{1}{m}.$$

Combining these two inequalities yields

$$\left| \frac{F^n(0)}{n} - \frac{F^m(0)}{m} \right| < \frac{1}{n} + \frac{1}{m}.$$

Since n and m were arbitrary, we have that $\{\frac{F^n(0)-0}{n}\}$ is a Cauchy sequence. Since $\mathbf{R}$ is complete this sequence must converge. Finally, since $\rho(f)$ is independent of x, the rotation number exists for all $x \in \mathbf{R}$.