Exercise 12 solution: We need to show $\rho(f) = p/q$ implies $F$ has a $p/q$-periodic point. Suppose $F$ has no $p/q$-periodic point. Then for all $x \in \mathbf{R}$, $F^q(x) \neq x + p$. By the Intermediate Value Theorem applied to the continuous function $F^q(x) - x$, either $F^q(x) > x + p$ for all $x \in \mathbf{R}$, or $F^q(x) < x + p$ for all $x \in \mathbf{R}$. Suppose $F^q(x) > x + p$ for all $x \in \mathbf{R}$. As in the solution of Exercise 11, when considering $F^q(x) - x$ it suffices to take $x \in [0, 1]$. By the Extreme Value Theorem $F^q(x) - x$ assumes a minimum value $m$ on $[0, 1]$. By assumption, $p < m$. Let $\epsilon = m - p$, so that $F^q(x) \geq x + p + \epsilon$ for all $x \in \mathbf{R}$. Since $F^q$ is increasing,

$$F^{2q}(x) \geq F^q(x + p + \epsilon) \geq (x + p + \epsilon) + p + \epsilon = x + 2(p + \epsilon),$$

and continuing, $F^{kq}(x) \geq x + k(p + \epsilon)$ for $k$ a positive integer. Hence

$$\rho(f) = \lim_{k \to \infty} \frac{F^{kq}(x) - x}{kq} \geq \frac{p + \epsilon}{q} > \frac{p}{q},$$

a contradiction.

Similarly, if $F^q(x) - x < p$ for all $x \in \mathbf{R}$, there is an $\epsilon > 0$ such that $F^q(x) \leq x + p - \epsilon$ for all $x \in \mathbf{R}$. Then $F^{kq}(x) \leq x + k(p - \epsilon)$, implying $\rho(f) \leq (p - \epsilon)/q < p/q$.

We have shown that if $\rho(f) = p/q$ then $F$ has a $p/q$-periodic point.