Exercise 13 solution: First note that if $G : \mathbf{R} \to \mathbf{R}$ is continuous and $\lim_{n\to\infty} G^n(x) = x_0$, then

$$x_0 = \lim_{n\to\infty} G^n(x) = \lim_{n\to\infty} G^{n+1}(x) = \lim_{n\to\infty} G(G^n(x)) = G(x_0),$$

so that x_0 is a fixed point of G.

Now let $x \in \mathbf{R}$. If $G(x) = x$ then $\lim_{n\to\infty} G^n(x) = x$. Suppose $G(x) \neq x$.

(i) If $G(x) < x$ then, since G is strictly increasing, $G^2(x) < G(x)$, $G^3(x) < G^2(x)$, and continuing, $G^{n+1}(x) < G^n(x)$ for all $n \geq 0$. Applying the Monotone Convergence Theorem to the decreasing sequence $\{G^n(x)\}$, either $G^n(x) \to -\infty$ as $n \to \infty$, or $G^n(x) \to x_0$ as $n \to \infty$. In the latter case x_0 is a fixed point of G as shown above.

(ii) If $G(x) > x$, then the sequence $\{G^n(x)\}$ is increasing. Hence either $G^n(x) \to \infty$ as $n \to \infty$, or $G^n(x) \to x_0$ as $n \to \infty$, and we are done.