Exercise 14 solution: Suppose $\rho(f) = p/q$. By Theorem 1, F has a p/q-periodic point x_0 which satisfies $F^q(x_0) - p = x_0$. Note that, for any integer k, $F^q(x_0 + k) - p = F^q(x_0) + k - p = x_0 + p + k - p = x_0 + k$, so $x_0 + k$ is a p/q-periodic point for all integers k. Set $G(x) = F^q(x) - p$, so that fixed points for G are p/q-periodic points for F.

Let $x \in \mathbf{R}$. Pick $k \in \mathbf{Z}$ with $x_0 + k \leq x < x_0 + k + 1$. If x is a fixed point for G then x is a p/q-periodic point for F. So suppose $G(x) \neq x$.

(i) If $G(x) < x$ then, since G is strictly increasing, $\{G^n(x)\}$ is a decreasing sequence. Suppose $G^n(x) < x_0 + k$ for some n; let n_0 be the smallest such n. Then $G^{n_0}(x) < x_0 + k$ and $G^{n_0-1}(x) \geq x_0 + k$. Since G preserves the ordering of points in $\mathbf{R}$, $x_0 + k = G(x_0 + k) \leq G(G^{n_0-1}(x)) = G^{n_0}(x)$, a contradiction. Hence $\{G^n(x)\}$ is bounded below by x_0+k and must converge to a point y by the Monotone Convergence Theorem. By Exercise 13, $G(y) = y$, so that y is a p/q-periodic point for F.

(i) If $G(x) > x$ then $\{G^n(x)\}$ is an increasing sequence. If, for some n, $G^n(x) > x_0 + k + 1$, then let n_0 be the smallest such n. We have $G^{n_0}(x) > x_0 + k + 1$ and $G^{n_0-1}(x) \leq x_0 + k + 1$. Since G is increasing, $x_0 + k + 1 = G(x_0 + k + 1) \geq G(G^{n_0-1}(x)) = G^{n_0}(x)$, a contradiction. Hence $\{G^n(x)\}$ is bounded above by x_0+k+1 and, as in (i), must converge to a point y satisfying $G(y) = y$, which is a p/q-periodic point for F.