Exercise 15 solution: We are given $hf(\theta) = gh(\theta)$ for all $\theta \in \mathbf{T}$, where h is a homeomorphism.

(a) We prove by induction: $hf^n(\theta) = g^n h(\theta)$ for all $\theta \in \mathbf{T}$ and all $n \geq 0$. Let $\theta \in \mathbf{T}$. The $n = 0$ case yields $h(\theta) = h(\theta)$. Suppose $hf^n(\theta) = g^n h(\theta)$ for some $n > 0$. Then $hf^{n+1}(\theta) = hf^n(f(\theta)) = g^n h(f(\theta)) = g^n gh(\theta) = g^{n+1}h(\theta)$, and we are done.

Note that $hf = gh$ implies $g^{-1}h = hf^{-1}$, so that f^{-1} and g^{-1} are also conjugate. Let $n > 0$. Since f^{-n} is the nth iterate of f^{-1}, we have $hf^{-n}(\theta) = g^{-n}h(\theta)$ by the argument above. Thus $hf^n(\theta) = g^n h(\theta)$ for all $n \in \mathbf{Z}$.

Let $f^n(\theta) \in o(\theta, f)$. Then $hf^n(\theta) = g^n h(\theta) \in o(h(\theta), g)$, so $h(o(\theta, f)) \subset o(h(\theta), g)$. Let $g^n h(\theta) \in o(h(\theta), g)$. Then $g^n h(\theta) = hf^n(\theta) \in h(o(\theta, f))$, so $o(h(\theta), g) \subset h(o(\theta, f))$. Hence $h(o(\theta, f)) = o(h(\theta), g)$.

(b) Suppose $o(\theta, f)$ is dense in $\mathbf{T}$. Let I be an open interval (arc) in $\mathbf{T}$. Since h is a homeomorphism, $h^{-1}(I)$ is an open set in $\mathbf{T}$. Pick $n \in \mathbf{Z}$ with $f^n(\theta) \in h^{-1}(I)$. Then $hf^n(\theta) = g^n h(\theta) \in I$, and $o(h(\theta), g)$ is dense in $\mathbf{T}$.

For the other direction, note that g is conjugate to f via the homeomorphism h^{-1}, so $o(\theta, g)$ dense in $\mathbf{T}$ implies $o(h^{-1}(\theta), f)$ is dense in $\mathbf{T}$.