Exercise 20 solution:

(a) By Exercise 19, $H(x+k) = H(x) + k$ for all integers k (see the solution to Exercise 6(b)). Suppose $H(J) = \{y_0\}$ for some interval $J \subset \mathbf{R}$. Let $i, j \in \mathbf{Z}$. By Proposition 3, $H(F^i(J)+j) = HF^i(J)+j = R_\alpha^i H(J)+j = R_\alpha^i(y_0)+j = y_0+i\alpha+j$, a real number. Hence H is constant on $F^i(J)+j$ for all integers i and j.

(b) Let K equal the union of all open intervals on which H is constant. By definition K is open. If K is not dense in $\mathbf{R}$, there exist $x \in \mathbf{R}$ and $\epsilon > 0$ such that $K \cap (x-\epsilon, x+\epsilon) = \emptyset$. Since H is increasing, if $x_1, x_2 \in (x-\epsilon, x+\epsilon)$, $x_1 < x_2$, then $H(x_1) < H(x_2)$. That is, $H((x-\epsilon, x+\epsilon))$ is an open interval V. If, for some integers i and j, $F^i(J)+j \cap (x-\epsilon, x+\epsilon) \neq \emptyset$, then H would be constant on an open interval in $(x-\epsilon, x+\epsilon)$, a contradiction. Hence $F^i(J)+j \cap (x-\epsilon, x+\epsilon) = \emptyset$ for all $i, j \in \mathbf{Z}$. This implies $R_\alpha^i(H(J))+j \cap V = \emptyset$ for all i and j, contradicting the fact $\{ R_\alpha^i(H(J))+j : i, j \in \mathbf{Z} \}$ is dense in $\mathbf{R}$. Thus K is an open, dense subset of $\mathbf{R}$.