Exercise 23 solution:

(a) Let $\omega_1 < \omega_2$. Then for all $x \in \mathbf{R}$, $F_{\omega_1}(x) < F_{\omega_2}(x)$. We claim $F^n_{\omega_1}(x) < F^n_{\omega_2}(x)$ for all $x \in \mathbf{R}$ and all $n \geq 1$. The $n = 1$ case is given above. Suppose $F^n_{\omega_1}(x) < F^n_{\omega_2}(x)$ for all $x \in \mathbf{R}$ and some $n > 1$. Then $F^{n+1}_{\omega_1}(x) = F_{\omega_1}(F^n_{\omega_1}(x)) < F_{\omega_1}(F^n_{\omega_2}(x))$ since F_{ω_1} is strictly increasing. And $F_{\omega_1}(F^n_{\omega_2}(x)) < F_{\omega_2}(F^n_{\omega_2}(x)) = F^{n+1}_{\omega_2}(x)$ by the inductive hypothesis.

(b) $F_0(x) = x + \frac{\epsilon}{2\pi}\sin(2\pi x)$. Since $F_0(0) = 0$, $F^n_0(0) = 0$ for all $n \geq 1$, and

$$\rho(0) = \lim_{n\to\infty} \frac{F^n_0(0) - 0}{n} = 0.$$

We now show that for δ sufficiently small, $F_{1-\delta}$ has a $1/1$-periodic point, i.e., $F_{1-\delta}(x) = x + 1$ for some $x \in \mathbf{R}$. Let $\delta < \epsilon/2\pi$, so that $\sin(2\pi x) = \delta 2\pi/\epsilon$ has a solution $x = x_0$. Then $F_{1-\delta}(x_0) - x_0 - 1 = -\delta + \frac{\epsilon}{2\pi}\sin(2\pi x_0) = 0$, so that x_0 is a $1/1$-periodic point for $F_{1-\delta}$. Thus $F_{1-\delta}$ has a $1/1$-periodic point for all $0 < \delta < \epsilon/2\pi$.

(c) Let $\epsilon > 0$ and $\omega_0 \in [0, 1)$ be given. Pick $n \in \mathbf{Z}$ such that $2/n < \epsilon$. Pick $k \in \mathbf{Z}$ satisfying $k - 1 < F^n_{\omega_0}(0) < k + 1$. Using the fact $F^n_{\omega_0}$ is strictly increasing (as in the solution of Exercise 12), $m(k-1) < F^{mn}_{\omega_0}(0) < m(k+1)$ for all m. Since $F^n_\omega(0)$ is a continuous function of ω, we can pick $\delta > 0$ such that $|\omega - \omega_0| < \delta$ implies $k - 1 < F^n_\omega(0) < k + 1$. Hence $m(k-1) < F^{mn}_\omega(0) < m(k+1)$ for all m, and we have $|F^{mn}_\omega(0) - F^{mn}_{\omega_0}(0)| < 2m$. Hence

$$\left|\frac{F^{mn}_\omega(0) - 0}{mn} - \frac{F^{mn}_{\omega_0}(0) - 0}{mn}\right| < \frac{2}{n},$$

yielding $|\rho(\omega) - \rho(\omega_0)| < \epsilon$. Thus $\rho(\omega)$ is continuous on $[0, 1)$.