Exercise 3 solution: Let n be an integer. Suppose there exist $x_1, x_2 \in [n, n+1)$ with $F(x_1) = F(x_2)$. Given F is a lift of f, $\pi F(x_1) = \pi F(x_2)$ implies $f\pi(x_1) = f\pi(x_2)$. Since f is injective, $\pi(x_1) = \pi(x_2)$, implying $x_1 = x_2 + k$ for some integer k. Given $x_1, x_2 \in [n, n+1)$, $k = 0$, and we have $x_1 = x_2$. Hence F is injective on $[n, n+1)$ for all integers n.

Suppose there exists an integer n such that F is strictly decreasing on $[n, n+1)$. Pick $x_1, x_2, x_3 \in [n, n+1)$, $x_1 < x_2 < x_3$, and an integer m satisfying $m < F(x_3) < F(x_2) < F(x_1) < m+1$. Note that, moving counterclockwise on $\mathbf{T}$ starting at $\pi(x_1)$, we encounter $\pi(x_2)$ before $\pi(x_3)$. However, moving counterclockwise on $\mathbf{T}$ starting at $\pi F(x_1) = f\pi(x_1)$, we encounter $\pi F(x_3) = f\pi(x_3)$ before $\pi F(x_2) = f\pi(x_2)$. Now, since f is orientation preserving, starting at $f\pi(x_1)$ and moving counterclockwise we encounter $f\pi(x_2)$ before $f\pi(x_3)$, a contradiction. Thus F is strictly increasing on $[n, n+1)$ for all integers n.

We remark that F is strictly increasing on $[n, n+1]$ for all integers n. Let $x \in [n, n+1)$, and pick a sequence $\{x_k\} \in [n, n+1)$ with $x_k < x_{k+1}$ for all k and $\lim_{k\to\infty} x_k = n+1$. Then $F(x) < \lim_{k\to\infty} F(x_k) = F(n+1)$ by the continuity of F at $n+1$. In particular, $F(n+1) > F(n)$, and so $F(m) > F(n)$ for any integer $m > n$.

We now show F is strictly increasing on $\mathbf{R}$. Let $x_1 < x_2$, with $x_1 \in [n, n+1)$ and $x_2 \in [m, m+1)$ for some integers $n \leq m$. If $n = m$ then $F(x_1) < F(x_2)$. If $n < m$ then $F(x_1) < F(n+1) \leq F(m) \leq F(x_2)$, and we're done.