Exercise 6 solution:

(a) Let $x \in \mathbf{R}$. Then $\pi F(x+1) = f\pi(x+1) = f\pi(x) = \pi F(x)$, so $F(x+1) = F(x)+k$ for some integer k. Since F is strictly increasing, $k \geq 1$. Suppose $k > 1$, so that $[F(x), F(x)+2] \subset F([x, x+1])$. Let $y \in (F(x), F(x)+1)$. By the Intermediate Value Theorem and the fact F is strictly increasing, there exist $x_1, x_2 \in (x, x+1)$, $x_1 < x_2$, with $F(x_1) = y$ and $F(x_2) = y+1$. Note that $\pi(x_1) \neq \pi(x_2)$. We have $\pi F(x_1) = \pi(y)$ and $\pi F(x_2) = \pi(y+1) = \pi(y)$, implying $f\pi(x_1) = \pi(y) = f\pi(x_2)$, contradicting the fact f is injective. Hence $k = 1$, and $F(x+1) = x+1$ for all reals x.

(b) We show by induction that for all reals x and for all positive integers k, $F(x+k) = F(x)+k$. This is clearly true when $k = 0$. Suppose $F(x+k) = F(x)+k$ for some $k > 0$. Then $F(x+k+1) = F(x+k)+1 = F(x)+k+1$, and we're done.

Now let $k < 0$ be an integer. Since $-k > 0$, we have $F(x) = F(x+k-k) = F(x+k)-k$, or $F(x+k) = F(x)+k$. Thus $F(x+k) = F(x)+k$ for all integers k.

Let n be an integer, and set $H(x) = F^n(x)$ and $h(\theta) = f^n(\theta)$. Let k be an integer. Since H is a lift of h by Exercise 4, $F^n(x+k) = H(x+k) = H(x)+k = F^n(x)+k$.