Math 345 - Problem Set 3 - due Friday, March 23

These will be graded on both correctness and clarity, so write careful, organized solutions (a good idea to keep in mind is that another person from the class should be able to read and understand your solution). All of your answers must be justified. **Honor Code:** For this and all problem sets, you are encouraged to work on solving these problems in groups. However, you MUST write up your solutions individually; in particular, you may not look at someone else's write-up. In addition, you must indicate who you worked with.

1. *The Two-envelope paradox* (loosely from Cover and Thomas) Suppose our game show host has two envelopes, one with twice the amount of money as the other. She randomly hands you one of the two, you open it, and then she asks whether you'd like to keep the money or switch and take the contents of the other envelope. A quick calculation shows that you expect to gain 25% be switching. So should you always switch? Does that strategy make any sense? What if you switch without even looking at the first envelope, and then she offers to let you switch back? You expect to gain 25% by switching back, so shouldn't you? How can this be?

   Let's be precise. Let $b > 0$ be fixed (by the producers of the game show; you don't know b), so that the two envelopes contain b dollars and $2b$ dollars. Let X_1 be the envelope you are given first, and let X_2 be the other envelope, so

   $$P(X_1 = b, X_2 = 2b) = P(X_1 = 2b, X_2 = b) = \frac{1}{2}.$$

   (a) Show that $E[X_1] = E[X_2] = \frac{3}{2}b$. If you either always stick with your envelope X_1 or you always switch to X_2, you expect to make $\frac{3}{2}b$ dollars.

   (b) Show that $E\left[\frac{X_2}{X_1}\right] = \frac{5}{4}$. If you switch, you expect to gain 25%. This is the source of the paradox. The problem is that

   $$E[X_2] \neq E[X_1] \cdot E\left[\frac{X_2}{X_1}\right].$$

   The value we care about is $E[X_2]$, not $E\left[\frac{X_2}{X_1}\right]$.

There are two types of people in this world
Type 1: Those who tend to take more risks when more money is on the table and
Type 2: Those who tend to take less risk and "lock in their gains".

Let $g_1(x)$ and $g_2(x)$ represent the respective gut reaction of those people upon opening the first envelope and seeing x dollars. The $g_i(x)$ are the probabilities that that person will switch, given that the first envelope contains x dollars. All we require is that $g_1(x)$ is a strictly increasing function (if $x > y$, then $g_1(x) > g_1(y)$, i.e., a Type 1 person takes more risks the larger x is) and $g_2(x)$ is a strictly decreasing function (and, of course, $0 \le g_i(x) \le 1$ since they are probabilities). Let Y_i be the random variable indicating whether the person of type i will **K**eep the first envelope or **S**witch, i.e., $Y_i = S$ with probability $g_i(X_1)$ and $Y_i = K$ with probability $1 - g_i(X_1)$.

(c) Show that $I(X_1; Y_i) > 0$ for both $i = 1, 2$, i.e., both people's guts know something (have some information about which envelope is which). Hint: write it as

$$H(X_1) - H(X_1 \mid Y_i) = H(X_1) - E_{y \in \{K,S\}}[H(X_1 \mid Y_i = y)];$$

all you should need about g_i is that $g_i(b) \neq g_i(2b)$.

(d) Which type of person should you be? Let Z_i be the amount of money that the person of type i receives when they keep or switch based on Y_i. Show that for one type of person, $E[Z_i] > \frac{3}{2}b$, no matter what b is, but for the other type of person $E[Z_i] < \frac{3}{2}b$. Both people's guts have information about which envelope is which, but one should listen to his gut and the other should do the opposite of what it tells him.

2. Let Q' be the discrete memoryless noisy channel from $X = (x_1, x_2) \in \{0,1\}^2$ to $Y = (y_1, y_2) \in \{0,1\}^2$, such that, for each of the two bits x_1 and x_2 sent, the chance of it being received wrong is $\frac{1}{2}$. In other words,

$$P(y_i = 1 \mid x_i = 0) = P(y_i = 0 \mid x_i = 1) = \frac{1}{2},$$

for $i = 1, 2$, and whether the second bit is received wrong is independent of whether the first is. We saw in class that the capacity of this channel is 0. If we want to encode and send one bit (either a 0 or a 1) through this channel (i.e., $N = 2$, $K = 1$ in the language we've been using in class), then the maximum probability of error, p_{BM}, is at least $\frac{1}{2}$.

Now suppose we have a channel Q from $X = (x_1, x_2) \in \{0,1\}^2$ to $Y = (y_1, y_2) \in \{0,1\}^2$ whose noise comes in "spurts" (has memory). Let the chance that the first bit is received wrong be $\frac{1}{2}$, i.e.,

$$P(y_1 = 1 \mid x_1 = 0) = P(y_1 = 0 \mid x_1 = 1) = \frac{1}{2}.$$

Given that the first bit was received wrong, the chance that the second bit is also received wrong is $\frac{3}{4}$. On the other hand, given that the first bit was received correctly, the probability that the second bit is also received correctly is $\frac{3}{4}$.

(a) Show that the probability that the second bit is received wrong is $\frac{1}{2}$.

(b) Let X be distributed uniformly among $\{00, 01, 10, 11\}$. Compute $I(X;Y)$ (It will be positive!).

(c) Suppose we want to encode and send one bit (either a 0 or a 1) through this channel ($N = 2$, $K = 1$). Find a way to encode and decode such that the maximum probability of error, p_{BM}, is $\frac{1}{4}$.