Chapter 4: the Earth's Motion and the Seasons

One cannot consider solar energy without becoming aware of the earth's motion, both its rotational motion about its own axis, and its orbital motion about the sun. These two modes of motion give rise to day and night, the number of hours of daylight, the angle of the sun in the sky, and the seasons. All of these are very important for understanding solar energy.

We start by assuming that the earth's orbit about the sun is a perfect circle of radius r = 1.50 x 10^{11} m. The actual orbit is slightly ecliptic, but the distance does not vary by more than 2% from this. We also begin by ignoring the tilt of the earth's axis with respect to the orbital plane (called the ecliptic). This, of course, ignores the effects of seasons. We will take care of this later. Here we consider only geometry factors and ignore atmospheric effects.

1. The Effect of Tilted Angle

Before discussing the earth's motion we first consider how the angle of the sun affects things. Suppose light with intensity I (in W/m^2) is incident on an area A. If the light rays are perpendicular to the area then the power incident upon the area is I = P A. The situation is illustrated on the left in the figure below. If, on the other hand, the light rays are not perpendicular to the area but, say, come at an angle θ to the normal, then less power is incident on the area. Using trigonometry we find that the incident power is given by

$$P = IA\cos\theta .$$

This situation is illustrated on the right in the figure below. The tilted area is actually twice that of the "untilted" area, but because it is tilted at an angle of 60°, the shadow cast by the two areas is exactly the same as is the total solar power incident on each. One way to think about this is that the "effective" area of the absorber is not the total area, but rather, is

$$A_{eff} = A\cos\theta .$$

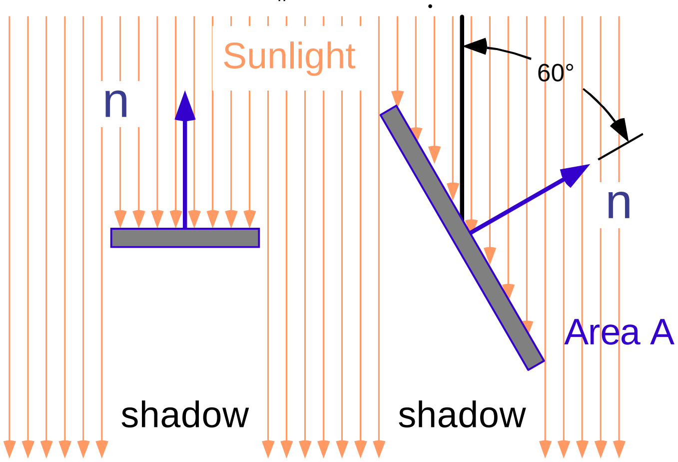

Figure 1. Drawing illustrating the shadow cast by two different areas. The area on the right is twice that of that on the left but, because it is tilted at an angle θ = 60° with respect to the rays, the solar power incident on both areas is the same.

So, the solar power incident on an object depends on its area A, the intensity of the sunlight I, and the tilt angle θ of A with respect to the sun's rays. This is important when considering the amount of solar power available at different times of day.

2. the Earth's Rotation

We all know that the earth rotates about its axis undergoing one revolution every 24 hours, or at least very close to that. The earth also moves around the sun at a rate of 1 trip every 365.25 days. If the earth's rotational axis were perpendicular to its orbital plane, called the **ecliptic**, the situation would look like that shown below.

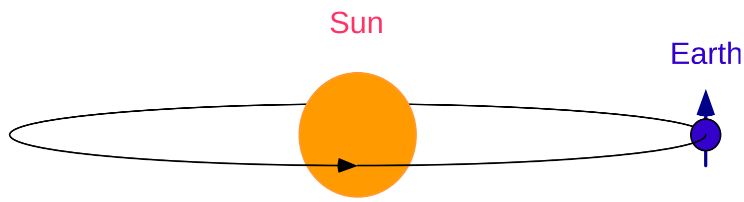

Figure 2. Simple view of the earth's orbit around the sun which ignores the tilt of the earth's axis with regard to the orbital plane (ecliptic).

Consider a point on the equator. The figure below shows the angle with which the sun's rays makes with the zenith (vertical direction) during different times of the day. When the sun is directly overhead the **zenith angle** (Z) is zero and the full intensity of sunlight, I_0 = 1380 W/m^2 (AM0 radiation) is directed upon the earth's surface.

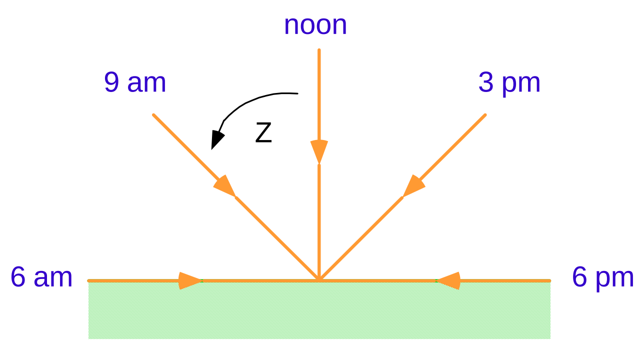

Figure 3. Drawing showing the zenith angle of the sunlight at various times of day at the equator, ignoring the tilt of the earth's axis.

The zenith angle varies linearly in time from –90° at sunrise (6 am) to +90° at sundown (6 PM). The solar power incident on an area A of the earth's surface then depends on Z, and is given by[1]

$$P(Z) = I_0 A \cos Z .$$

Now the zenith angle is a simple function of the time of day. In radians it is given by

$$Z(t) = \begin{cases} 0 & t < -6hr \\ 2\pi \dfrac{t}{24hr} & -6hr < t < 6hr \\ 0 & t > 6hr \end{cases},$$

where the daily zero of time is taken to be noon (e. g., 6 a. m. is given by t = –6 hr). A graph of the time-variation in the solar power on a unit area of the earth's surface at the equator is shown as the solid red curve in figure below. The solid green curve is the same plot for Oberlin which has a latitude 41°N of the equator.

[1] This accounts only for geometry and, as mentioned earlier, ignores the tilt of the earth's axis. Moreover, it ignores the effects of the atmosphere. These will be considered in a subsequent chapter.

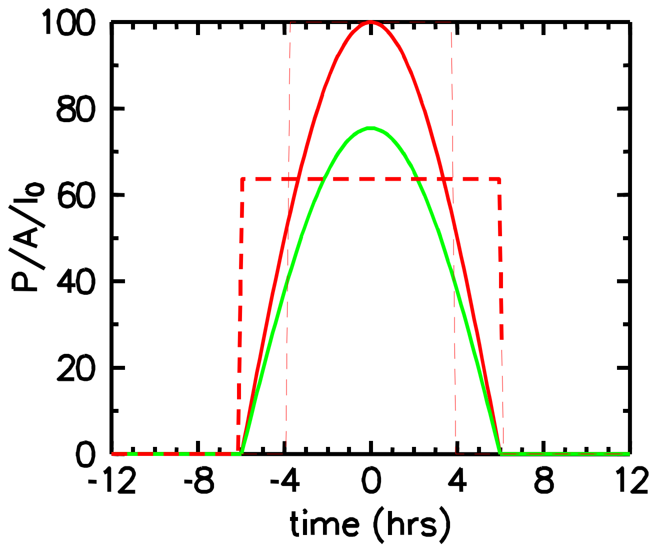

Figure 4. Graph of the power per unit area incident on the earth's surface versus time of day, neglecting the tilt of the earth's axis with respect to the ecliptic (or at the vernal or autumnal equinoxes). The red curve is calculated for the equator while the green curve is for a latitude of 41°N (Oberlin).

The total solar energy per unit area incident on the earth's surface at the equator during the entire day is obtained by finding the area underneath the solid red curve. This area is 763%-hr. The two dashed cuves above represent rectangles having the same area. The total energy is the same as if the sun stood still overhead providing 100% intensity for exactly 7.63 hours. This is represented by the "taller" of the two rectangles. Alternately, the total energy is as if the sunlight had a reduced intensity of 63.6% for a total of 12 hours. This is represented by the "wider" of the two rectangles. Using $I_0 = 1.38$ kW/m^2 we see that, a total energy of 10.5 kW-hr/m^2 is deposited during the entire day.

2. Latitudes Away from the Equator

If the earth's axis has no tilt then, as the earth rotates, every point on the earth's surface receives exactly 12 hours of daylight and 12 hours of darkness. What makes other **latitudes** different from the equator is that the sun is not straight overhead at noon. In the northern hemisphere at a latitude L, after the sun rises it travels across the southern sky reaching its maximum altitude (angle measured from the horizon) at noon, 90°–L. After noon the sun continues across the southern sky, setting due east, 12 hours after sunrise. The angle between the sunlight and zenith for a latitude of 41°N is shown in the drawing below.

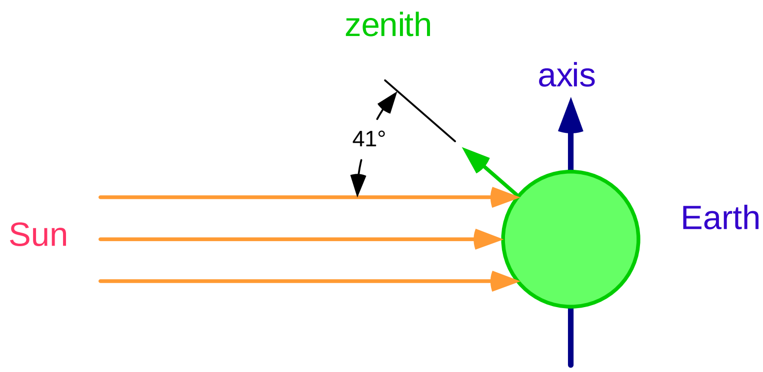

Figure 5. Sketch indicating the elevation angle of the sun at noon for latitude 41°N.

Thus, for all latitudes, the graph of the solar flux versus time through the day looks similar to what occurs at the equator, but with one key difference. At noon, the intensity is not the same as at the equator because the sun never reaches directly overhead. Instead, the maximum intensity is given by

$$I_{\max} = I_0 \cos L .$$

Thus, at Oberlin's latitude, 41°N, the solar flux reaches a maximum of 1040 W/m^2, 25% lower than at the equator. The full time-dependence of the flux is plotted as the green curve in the above figure. It is obtained by combining the above two equations to give

$$P = I_0 A \cos(Z) \cos(L) .$$

The total energy deposited in a full day is reduced similarly to 7.9 kW-hr/m^2. The actual annual daily energy deposited in our local about half this amount due to seasonal effects, the atmosphere, and local weather conditions.

3. Tilt of the Earth's Axis -- the Seasons

In reality, things are a little more complicated than described above. The earth's rotational axis is actually tilted at an angle of 23.5° away from the perpendicular to the ecliptic. In other words, Figure 1, which shows the earth's axis being "straight up" is not correct. As the earth moves around the sun, the angle between the earth's axis and a line drawn to the sun varies. At the **winter solstice** (approximately December 22) the axis tilts away from the sun so that, in the northern hemisphere, the sun sits even lower in the sky and the days are shorter than 12 hours. The sun does not rise in the (due) east and set in the (due) west but, instead, rises south of east, travels low across the southern sky, then sets south of west. At the **summer solstice** (approximately June 22) the earth tilts towards the sun (66.5°) so that, in the northern hemisphere, the days are longer and the sun rises higher in the sky. At this time the sun rises north of east and sets north of west. The days are longer than 12 hours. At either the **vernal equinox** or the **autumnal equinox**, the tilt of the earth's axis is oriented so that the angle between it and solar rays is exactly 90°. On these two days of the year the day is 12 hours long at all places on earth.

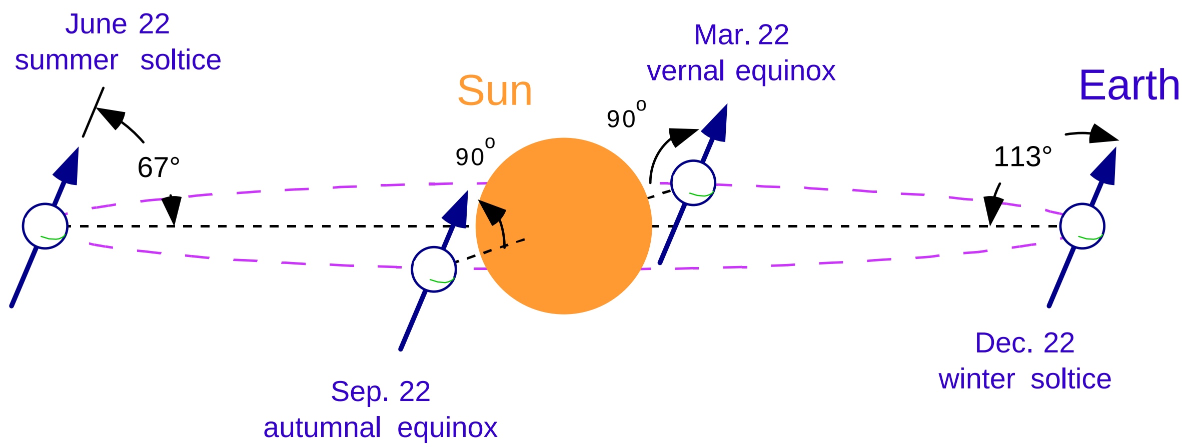

Figure 6. Orbit of the earth around the sun showing the orientation of the earth's axis at various times during the year.

Because of the tilt of the earth's axis, the conclusions of Sections 1 and 2 above are not generally correct. They are correct, however, during the two equinoxes when the tilt of the earth's axis is neither towards nor away from the sun. For other times of the year we need to modify our calculations to incorporate the tilt of the axis.

The next two sections are quite mathematical. You are not required to understand the details. Just be sure that you understand the results.

a) Tilt of Earth's Axis

To understand the seasons we must be able to find the angle D' between the earth's axis and the earth/sun line. This angle is called the **co-declination**. In the figure below the earth's axis is represented by the vector $\vec{a}$ and the earth-sun line is the vector $\vec{r}$, the position of the earth relative to the sun. The earth's orbit occurs in the x-y plane (known as the ecliptic) with the sun at the origin. The earth's axis lies in the y-z plane and is tilted at an angle β from the z-axis. The position of the earth relative to the vernal equinox is measured, in polar coordinates, by the angle α, which varies linearly in time,

$$\alpha(t) = \frac{2\pi t}{T_S},$$

where T_S = 365.25 days is the period of the earth's orbit around the sun. The above expresses the angle in radians. To express in degrees replace the 2π (radians) in the above expression with 360°.

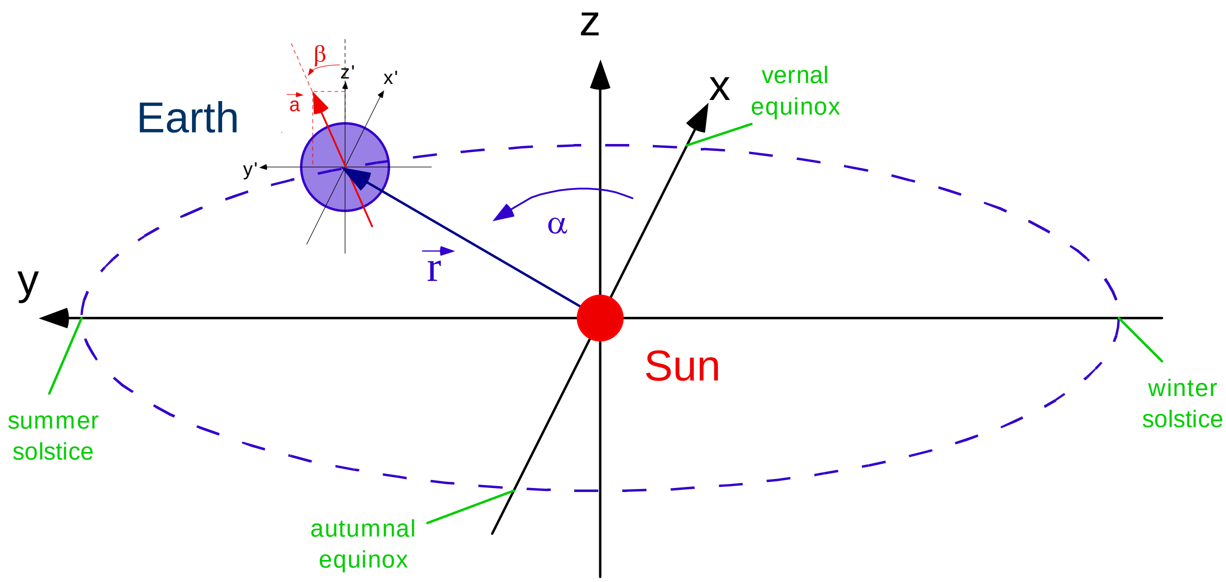

Figure 7. Coordinate system with sun at origin for determining the angle between the earth's axis and the earth-sun line, the co-declination D'.

The earth's axis is given by the vector

$$\vec{a} = \sin\beta\hat{y} + \cos\beta\hat{z},$$

where β = 23.5° is the tilt angle. The position of the earth is given by

$$\vec{r}(t) = r\{\cos\alpha(t)\hat{x} + \sin\alpha(t)\hat{y}\}.$$

The *dot product* between two vectors is the product of the length of each of the vectors times the cosine of the angle between them. The angle we are looking for is the angle between the vectors $\vec{a}$ and $-\vec{r}$. Thus, the angle D' is given by

$$-\vec{r}\cdot\vec{a} = |\vec{r}||\vec{a}|\cos D' = r\sin\alpha(t)\sin\beta,$$

or

$$\cos D' = \sin\beta\sin\left(\frac{2\pi t}{T_S}\right).$$

This equation allows us to solve for the co-declination D' given the time of year t, since both T_S and β are known.

b) Geocentric Coordinate System

A **geocentric** coordinate system is one centered on the earth. In such a system the sun is located by two angles, the co-declination D' and the **hour angle**, H. The geocentric system has the z-axis along the earth's rotational axis. The two other axes, x and y, are mutually perpendicular and lie in the equatorial plane. The x-axis is oriented towards some **local meridian** (the one which runs through Greenwich, England, for instance).

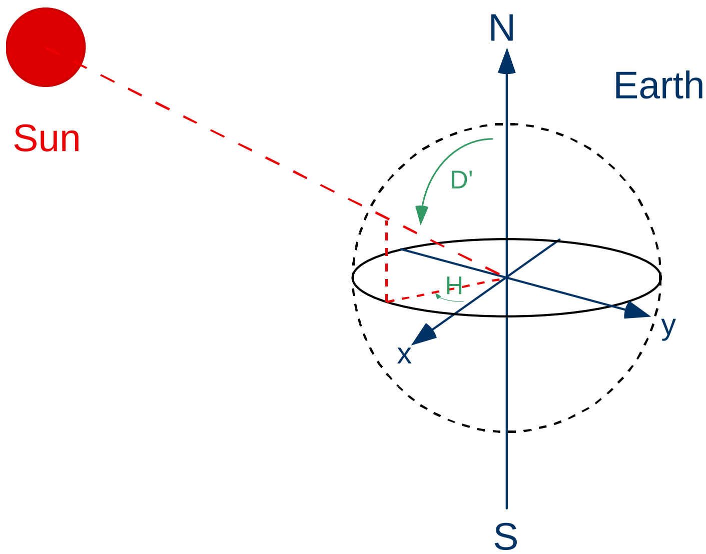

Figure 8. Figure shows a geocentric coordinate system.

It is convenient to refer to the location of the sun with respect to local coordinates on the earth's surface. Imagine a 3-d coordinate system with the x-axis pointing south, the y-axis pointing east, and z-axis pointing up (directly away from the earth's surface). The position of the sun is then determined by the zenith (Z) and azimuthal (A) shown in the figure below. The **zenith angle** is measured from the vertical. The **azimuthal angle** is measured from the x axis to a line given by the projection of the sun's ray in the x-y plane (the earth's surface).

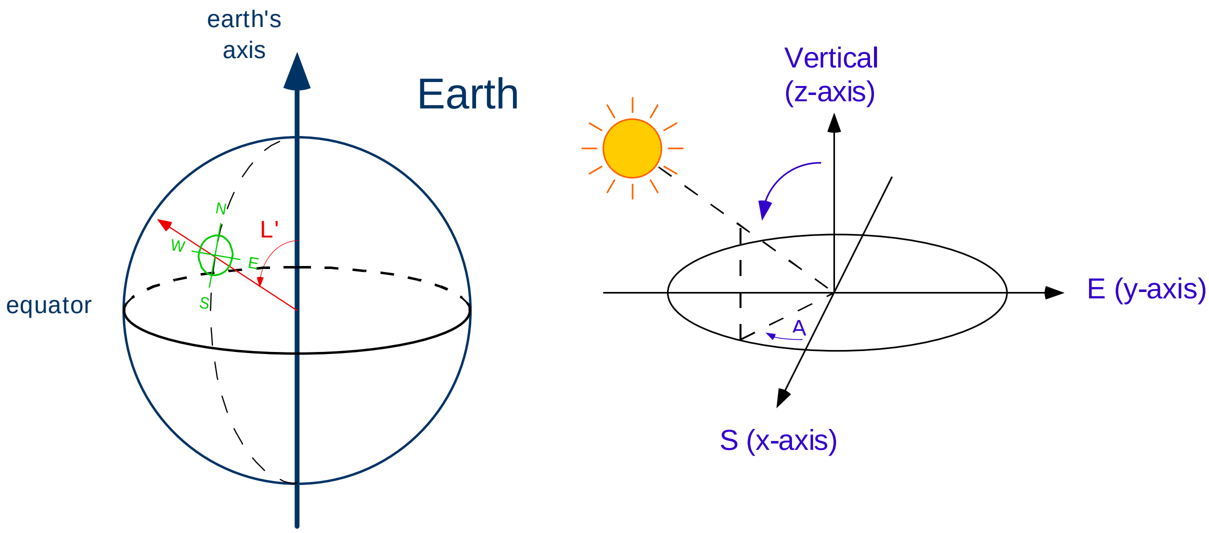

Figure 9. Local coordinate system attached to the earth's surface indicating the location of the sun with the zenith and azimuthal angles, Z and A. The zenith angle is zero when the sun is directly overhead. The azimuthal angle is the angle is the angle between the southern direction and the projection on the earth's surface of the line drawn from the origin to the sun.

At one of the equinoxes, at the equator, the sun rises in the east with Z = –90° and A = –90°. As the morning wears on, the zenith angle decreases but A stays the same -- due east. At noon the sun is directly overhead -- Z = 0 and A is undefined (neither east or west). After noon Z grows from 0 to +90° and A = +90°.

Well, that's not too bad. But now we need to a) look at other times of the year when the tilt of the earth's axis has an effect, and b) consider what occurs away from the equator. The equations that describe all this are very complicated.2 Nevertheless, you can see what happens intuitively and learn how to use the formula that tells us the zenith and azimuthal angles of the sun at any time of day, at any time of year.

Consider the winter solstice, again, at the equator. The earth's axis is tilted away from the sun so that the noonday sun appears directly overhead for locations having a latitude of 23.5° N. At the equator, the sun rises in the east, but reaches a noon zenith of 90°–23.5° = 71.5° .

4. Solar Coordinates

In order to design buildings that will be able to make maximum use of the sun's light and heat it is important to know exactly where the sun is located throughout the day and throughout the year. These are given by the zenith angle Z and the azimuthal angle A. In general, each will vary throughout the day, will vary with latitude, and will change with the seasons (i. e., will vary throughout the year). It is also important to know the time of sunrise and sunset, and the number of daylight hours. To find all of these quantities involves some very complex geometry and trigonometry. Here I simply summarize the equations. The first equation, for the co-declination D', has already been derived. Next, we need to know the hour angle, H, which is given, in radians, by

$$H(t) = \omega_E t = \frac{2\pi t}{T_E},$$

where T_E = 24 hrs. is the period of the earth's rotation about its axis, and t is the time of day measured from some standard reference point, say the meridian line that goes through Grenwich, England. So, t is GMT or universal time. The hour angle must then be adjusted for our specific longitude.

Once we have obtained the co-declination D', the latitude L, and the hour angle H, then the zenith angle is given by

$$\cos Z = \cos D' \cos L' + \sin D' \sin L' \cos H,$$

were L' = 90° – L is the co-latitude. The azimuthal angle is then given by

$$\tan A = \frac{\sin D' \sin H}{\sin D' \cos L' \cos H - \cos D' \sin L'}.$$

Sunrise occurs when the zenith angle is 90°. The hour-angle of sunrise, H_0 is given by

$$\cos H_0 = -\cot D' \cot L' = -\frac{\tan L}{\tan D'},$$

so that the times for sunrise and sunset (in hours before or after solar noon) are given by

$$t_0 = \pm \frac{H_0}{2\pi} T_E.$$

<u>Example 1:</u>
Let's apply the above equations to find the time for sunrise and sunset here in Oberlin on Sunday, Feb. 16.

Solution:

[2] See Sol Wieder, *An Introduction to Solar Energy and Scientists and Engineers* (John Wiley & Sons, 1992), pp. 19-37.

First, find the number of days since the winter solstice, Dec. 22. Feb. 16 is 16 days in Feb., plus 31 days in Jan., plus 9 days in Dec. past the winter solstice. The winter solstice is exactly (3/4) 365.25 days past the vernal equinox. Thus, Feb. 16 is 239 days after the summer solstice.

$$\alpha = 360° \times 330/365 = 325°.$$

The co-declination D' then is given by

$$D' = \cos^{-1}(\sin\alpha \sin\beta) = \cos^{-1}(\sin(325°) \sin(23.5°)) = 103°,$$

that is, we are still tilted away from the sun, but headed towards 90°.

Oberlin's latitude L = 41.3° so the co-latitude is 48.7°, thus the hour-angle for sunrise is

$$H_0 = \cos^{-1}(-\tan(41.3°) / \tan(103°)) = 78.2°$$

The time for sunrise is then

$$t_0 = (-78.2°/360°)\ 24\ \text{hrs} = -5.21\ \text{hrs.}$$ (before noon by 5.21 hours), or 6:48 AM

Similarly, sunset will be 48 minutes before 6 PM, or 5:12 PM

The length of the day is then

12 hrs - 2(48 min) = 10.42 hours or 10 hrs 25 min.

Incidentally, the local paper gives sunrise at 7:20 AM and sunset at 6:01 PM, both Eastern Daylight Savings time. The total daylight = 10.65 hours. The discrepancies between the calculated sunrise/sunset times and their actual values (as stated in the news paper) are due to our location relative to the standard meridian for our time zone. I do not know the origin yet of the difference in total daylight time (which doesn't depend on time zone stuff).

5. Summary

The above section was very mathematical. What are the important ideas? First, focus on the two days of the year when the tilt of the earth's axis may be ignored -- the *vernal* and *autumnal* equinoxes. On these two days the hourly variation in solar intensity is shown in Figure 1. Everywhere on earth the day is exactly 12 hours long. The solar power deposited on a horizontal surface area is maximum at noon. At the equator, the noon-day-sun is directly overhead while at other latitudes, the sun sits lower in the sky. In the northern hemisphere the sun rises due east, travels across the southern sky to its maximum zenith (equal to the latitude L) at noon. During the afternoon the sun continues across the southern sky until it sets due west.

During other times of the year the tilt of the earth's axis with respect to the ecliptic comes into play. On the winter solstice the axis is tilted away so that northern latitudes receive even less sunlight. Instead of reaching a maximum zenith at noon equal to the latitude, the zenith is L-23.5°. Moreover, the days are shorter than 12 hours. The sun rises south of east, travels low across the southern sky and sets south of west. In the northern hemisphere this corresponds to winter.

At the summer soltice the earth's axis is tilted towards the sun so that northern latitudes receive more sunlight. This corresponds to summer. The sun rises north of east, travels high across the southern sky reaching a noon zenith of L+23.5°, then continues across the southern sky, setting north of west. The days are longer than 12 hours.

There is a marvelous site on the world-wide-web which help you calculate the Sun's position at any time of the day. All you have to do is enter the latitude and longitude, the day of the year, and the time of day -- the solar position calculator does the rest. Connected with this position calculator are nice tutorials on the hour angle, altitude, and the azimuthal angle.

6. South Walls, Tilted Surfaces, and Tracking Systems

Earlier we have considered the amount of sunlight that is deposited on a horizontal surface. Of course we may interested in other surfaces. In the northern hemisphere it is important to consider the sunlight incident on a south-facing wall. The analysis is similar to that for a horizontal surface, but involves a different angle.

Solar cells are devices which convert sunlight into electricity. How solar cells are actually mounted has a big impact on the amount of energy they may produce. Horizontally oriented cells will receive the amount of sunlight we have discussed above. For northern latitudes the incident sunlight will be largest in the summer months. A better way to mount solar cells is at an angle equal to the latitude. Thus, at the equinoxes, the cells will be directed right at the noon day sun maximizing the energy they produce. Of course, the incident sunlight will be maximum at noon and lower at other times of the day. However, if the cells are mounted in such a way that their zenith angle may be adjusted it is possible to make sure they are always facing the noon day sun.

Tracking systems allow the cells to follow the sun during the day. A tracking system may orient the cell so that it is facing east when the sun rises and follows it as it travels west during the day. Such a tracking system is called a 1-axis system. If we combine this with the ability to adjust the zenith angle it becomes a 2-axis tracking system. Such a system allows maximum solar collection but is, of course, rather complicated.