Change in a cycle (*or* "There and back again")

(a) Isothermal change: $p_a V_a = p_b V_b = nRT_a$ so

$$\frac{p_b}{p_a} = \frac{V_a}{V_b} = \frac{1}{3.00} = 0.333$$

(b) Adiabatic change: $p_a V_a^\gamma = p_c V_c^\gamma$ so

$$\frac{p_c}{p_a} = \left(\frac{V_a}{V_c}\right)^\gamma = \frac{1}{(3.00)^{7/5}} = 0.215$$

(c) Adiabatic change: $T_a V_a^{\gamma-1} = T_c V_c^{\gamma-1}$ so

$$\frac{T_c}{T_a} = \left(\frac{V_a}{V_c}\right)^{\gamma-1} = \frac{1}{(3.00)^{2/5}} = 0.644$$

(d) For the isothermal expansion

$$W_1 = \int_{V_a}^{V_b} p\,dV = \int_{V_a}^{V_b} \frac{nRT_a}{V}\,dV = nRT_a \int_{V_a}^{V_b} \frac{dV}{V} = nRT_a \ln\left(\frac{V_b}{V_a}\right)$$

so

$$\frac{W_1}{nRT_a} = \ln\left(\frac{V_b}{V_a}\right) = \ln(3.00) = 1.10.$$

(e) For the isothermal expansion of an ideal gas there is no change in internal energy, so

$$\frac{Q_1}{nRT_a} = \frac{W_1}{nRT_a} = 1.10.$$

(f) For the isothermal expansion of an ideal gas there is no change in internal energy, $\Delta E_{\text{int},1} = 0$.

(g) For the isothermal expansion

$$\Delta S_1 = \int_i^f \frac{dQ}{T} = \frac{1}{T_a}\int_i^f dQ = \frac{Q_1}{T_a} \quad \text{so} \quad \frac{\Delta S_1}{nR} = \frac{Q_1}{nRT_a} = 1.10.$$

(h) For the adiabatic compression $pV^\gamma = p_a V_a^\gamma$ so

$$\begin{aligned} W_3 &= \int_{V_c}^{V_a} p\,dV = \int_{V_c}^{V_a} \frac{p_a V_a^\gamma}{V^\gamma}\,dV = p_a V_a^\gamma \int_{V_c}^{V_a} \frac{1}{V^\gamma}\,dV = p_a V_a^\gamma \left[-\frac{1}{(\gamma-1)V^{\gamma-1}}\right]_{V_c}^{V_a} \\ &= -\frac{p_a V_a^\gamma}{\gamma-1}\left[\frac{1}{V_a^{\gamma-1}} - \frac{1}{V_c^{\gamma-1}}\right] = -\frac{p_a V_a}{\gamma-1}\left[1 - \left(\frac{V_a}{V_c}\right)^{\gamma-1}\right] \end{aligned}$$

and

$$\frac{W_3}{nRT_a} = -\frac{1}{\gamma-1}\left[1 - \left(\frac{V_a}{V_c}\right)^{\gamma-1}\right] = -\frac{1}{2/5}\left[1 - \frac{1}{(3.00)^{2/5}}\right] = -0.889.$$

(i) For the adiabatic compression $Q_3 = 0$.

(j) For the adiabatic compression

$$\frac{\Delta E_{\text{int},3}}{nRT_a} = \frac{Q_3}{nRT_a} - \frac{W_3}{nRT_a} = 0.889.$$

(k) For the adiabatic compression $\Delta S_3 = 0$.

(ℓ) For the cooling at constant volume to reduce pressure, $W_2 = 0$.

(m) For the cycle $\Delta E_{\rm int} = 0$, i.e. $\Delta E_{\rm int,1} + \Delta E_{\rm int,2} + \Delta E_{\rm int,3} = 0$, but we know $\Delta E_{\rm int,1}$ from part (f) and $\Delta E_{\rm int,3}$ from part (j). For the cooling at constant volume to reduce pressure,

$$\frac{\Delta E_{\rm int,2}}{nRT_a} = -0.889.$$

(n) Using $\Delta E_{\rm int,2} = Q_2 - W_2$ for this cooling,

$$\frac{Q_2}{nRT_a} = -0.889.$$

(o) For the cycle $\Delta S = 0$, i.e. $\Delta S_1 + \Delta S_2 + \Delta S_3 = 0$, but we know ΔS_1 from part (g) and ΔS_3 from part (k). For the cooling at constant volume to reduce pressure,

$$\frac{\Delta S_2}{nR} = -1.10.$$

(p) The efficiency is

$$\frac{W_1 + W_2 + W_3}{Q_1} = 1 + \frac{W_3}{Q_1} = 0.192.$$

(q) The Carnot efficiency (maximum possible) is, from part (c),

$$1 - \frac{T_c}{T_a} = 0.356.$$

As required, the efficiency of this cycle is less than the Carnot efficiency.